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Formal Relationship 16

Historical Addendum to 'Life lived equals life left in stationary populations'

Joshua R. Goldstein

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Abstract

This note provides some earlier history of the relationship given in Formal Relationships 1, "Life left equals life lived in stationary populations," (Goldstein 2009) and shows that while the expectation of life at the mean age of the population is close to the mean age, this is not exactly so.

1. Earlier statements and proofs of the result

The first paper in the "Formal Relationships," special collection, 'Life left equals life lived in stationary populations," (Goldstein 2009) did not contain any reference to the originator of the relationships. However, Lotka (1939) stated the result and makes it clear it was already part of the general actuarial knowledge of the time.

Lotka[1] wrote

> *For a stationary population with an age distribution proportional to the life table, it is known that* ["on sait que"]
>
> $$\overline{e_x} = \bar{x}$$
>
> *that is, the mean of the life expectancies of all the individuals in the population is equal to their mean age.**
>
> ...
>
> **See for example E.F. Spurgeon.* Life Contingencies*, 1929, pp. 210-211.*

[1] This English passage is from the Smith and Rossert translation, pages 81-82. The French original appears on page 37 of Lotka (1939).

Lotka does not provide a derivation, but the source he cites (Spurgeon 1929) does provide a full proof. The use of "for example" suggests that the result was well-established. Indeed, Spurgeon's text was an update of the original actuarial textbook for the Institute of Actuaries written by King (1st edition 1887, 2nd edition 1902), which contains nearly the same derivation, noting "the total future lifetime of a stationary population is exactly equal to its total past lifetime" (p. 67, 1902 edition). The "textbook" nature of this publication suggests that the relationship was broadly known at the end of the 19th century. More extensive investigation of the historical actuarial literature may reveal earlier mentions.

2. On Lotka's approximation

In their translation of Lotka (1998, p. 82), Smith and Rossert provide the following "Translator's note," providing their own intuition on the result:

> The equality is explained by noting that the expected age of a random individual in the life table will be the mean age of the table, and the individual will have been sampled halfway through his or her lifetime.

To say that the "individual" aged $\bar{a}$ is halfway through his or her lifetime is approximately, but not exactly, true. It is equivalent to claiming, using parentheses rather than a subcript to indicate the age argument,

$$e(\bar{a}) = \bar{a}, \tag{1}$$

because then the expected age at death would be $e(\bar{a}) + \bar{a} = 2\bar{a}$ and the population age would be half of the expected life time length, $\bar{a} = (1/2)(e(\bar{a}) + \bar{a})$.

Indeed, Lotka extensively discusses (1), which although he uses the "equals" sign, the context makes clear is meant to be read as an approximation (1939, pages 31-37). He first shows, in a number of then-available life tables, that one can well approximate the mean age by finding the age at which remaining life expectancy equals itself. He then shows that (1) holds exactly when mortality follows De Moivre's linear survival curve.

However, (1) does not generally hold. Indeed, in a modern low-mortality life table, $e(\bar{a})$ is more than a year less than $\bar{e}$ (and its equivalent $\bar{a}$). For example, in Sweden, the combined-sex period life table of 2008 had a mean age of 41.7, but remaining life expectancy at this mean age was 40.5 years.

The *inequality* between average age and life expectancy at the average age follows from Jensen's inequality, which allows us to say that $\bar{e} > e(\bar{a})$, whenever $e(a)$ is convex in age, with the reverse being true if $e(a)$ is concave.

One can also use the standard Taylor series expansions of the moments of functions of random variables (e.g., Rice 1995, p. 149), to provide an explicit second order approximation. This gives

$$\bar{e} \approx e(\bar{a}) + \frac{e''(\bar{a})}{2}\sigma_a^2, \tag{2}$$

where $\bar{a}$ is the mean age of the stationary population, $e''(\bar{a})$ is the second derivative with respect to age of life expectancy evaluated at the mean age of the stationary population, and σ_a^2 is the variance of age of the stationary population.

In the case of De Moivre's survival curve, life expectancy is linear, and so Lotka's illustration gave him an exact relationship. In the life tables that Lotka examined, $e(a)$ was concave at early ages and convex at older ages, a fortunate coincidence of errors canceling one another to make the approximation quite accurate.

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