Response Letter to "Unobserved Population Heterogeneity: A Review of Formal Relationships", *Demographic Research*, September 2014

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November 7, 2019

In this Response Letter we prove a formula that was given as eq. (52*) in the original version of our article on "Unobserved population heterogeneity: A review of formal relationships" (Vaupel and Missov, 2014). In doing so, we also prove some related formulas.

We had learned, based on feedback from Joshua Goldstein and Cosmo Strozza, that our proof of eq. (52*) was inadequate. We then discovered that proof depended on a restrictive assumption. We corrected the article by deleting eq. (52*) and providing an alternative, more general expression, which we labeled eq. (39*). (In the article, we put an asterisk after an equation number if proof of the equation had not previously been published; we follow this practice in this Letter.)

The formula we deleted was:

$$\bar{\rho}(x,y) = \rho(x,y)\left[\bar{s}_c(x,y)\right]^{\gamma}\,, \tag{1*}$$

where $\bar{\rho}(x,y)$ is the observed rate of progress in reducing the force of mortality in a population cohort at age x and at some time y, $\rho(x,y)$ is the unobserved rate of progress in reducing the force of mortality for each and every individual in the population, $\bar{s}_c(x,y)$ is cohort survivorship to age x in year y, and γ is the squared coefficient of variation of gamma-distributed frailty. We failed to state the key assumption that

$$\rho(a, y - x + a) = \rho(x, y), \qquad 0 \le a \le x\,,$$

i.e., that the rate of progress on the individual level is constant over age for the individuals born at time $y - x$. Given this assumption in the context of gamma-distributed fixed frailty, eq. (1*) can be proven as follows.

Proof. Consider the basic fixed-frailty relationship

$$\bar{\mu}(x, y) = \bar{z}(x, y)\mu(x, y)\,, \tag{2}$$

where $\bar{\mu}(x, y)$ is the population force of mortality at age x in year y, $\mu(x, y)$ is the corresponding force of mortality for the standard individual, and $\bar{z}(x, y)$ denotes the average gamma-distributed frailty at age x of individuals born in $y - x$. Eq. (2) is the two-dimensional equivalent of eq. (16) in the article. Taking relative derivatives with respect to time and reversing signs leads to

$$\bar{\rho}(x, y) = \rho(x, y) - \frac{1}{\bar{z}(x, y)} \frac{\partial \bar{z}(x, y)}{\partial y}\,. \tag{3*}$$

Eq. (31) of the article provides an expression for $\bar{z}$ for fixed-frailty models when frailty is gamma distributed; the two-dimensional form of eq. (31) is

$$\bar{z}(x, y) = (1 + \gamma H_c(x, y))^{-1}\,. \tag{4}$$

where $H_c(x, y)$ is the cumulative hazard at age x for the cohort born at time $y - x$. Substituting (4) in (3*) leads to

$$\begin{aligned}\bar{\rho}(x, y) = \rho(x, y) + \frac{\gamma \frac{\partial H_c(x,y)}{\partial y}}{1 + \gamma H_c(x, y)} = \rho(x, y) + \frac{\gamma \int\limits_0^x \frac{\partial \mu(a, y-x+a)}{\partial y} da}{1 + \gamma H_c(x, y)} = \\ = \rho(x, y) - \frac{\gamma \int\limits_0^x \rho(a, y - x + a)\mu(a, y - x + a)da}{1 + \gamma H_c(x, y)}\,.\end{aligned} \tag{5}$$

Assuming $\rho(a, y - x + a) = \rho(x, y)$ for $0 \le a \le x$, yields

$$\int_0^x \rho(a, y-x+a)\mu(a, y-x+a)da = \rho(x,y)\int_0^x \mu(a, y-x+a)da = \rho(x,y)H_c(x,y)\,,$$

and consequently

$$\bar{\rho}(x,y) = \frac{\rho(x,y)(1+\gamma H_c(x,y)) - \gamma\rho(x,y)H_c(x,y)}{1+\gamma H_c(x,y)}\,. \tag{6}$$

Simplification of (6) gives

$$\bar{\rho}(x,y) = \frac{\rho(x,y)}{1+\gamma H_c(x,y)}\,. \tag{7*}$$

Substitution in (7*) of the two-dimensional version of eq. (33) of the article,

$$\bar{s}_c(x,y) = (1+\gamma H_c(x,y))^{-\frac{1}{\gamma}}\,, \tag{8}$$

completes the proof of (1*), the expression we set out to prove,

$$\bar{\rho}(x,y) = \rho(x,y)[\bar{s}_c(x,y)]^{\gamma}\,.$$

□

Extension. Suppose $\rho(x,y)$ varies with age as well as time. Then combining equations (5) and (8) leads to

$$\bar{\rho}(x,y) = \rho(x,y)[\bar{s}_c(x,y)]^{\gamma} + \frac{(\rho(x,y) - \tilde{\rho}(x,y))\,\gamma H_c(x,y)}{1+\gamma H_c(x,y)}\,, \tag{9}$$

where

$$\tilde{\rho}(x,y) = \frac{\int_0^x \rho(a, y-x+a)\,\mu(a, y-x+a)\,da}{\int_0^x \mu(a, y-x+a)\,da} \tag{10}$$

is the weighted average value of the rates of progress up until age x for the cohort born at time $y-x$. Then (9) can be rewritten as

$$
\begin{aligned}
\bar{\rho}(x,y) &= \rho(x,y)[\bar{s}_c(x,y)]^{\gamma} + \frac{(\rho(x,y) - \tilde{\rho}(x,y))\,(1+\gamma H_c(x,y) - 1)}{1+\gamma H_c(x,y)} \\
&= \rho(x,y)[\bar{s}_c(x,y)]^{\gamma} + (\rho(x,y) - \tilde{\rho}(x,y))(1 - [\bar{s}_c(x,y)]^{\gamma})\,. \qquad (11^*)
\end{aligned}
$$

If $\rho(x,y) - \tilde{\rho}(x,y) \approx 0$, which might be the case at older ages, or $1 - [\bar{s}_c(x,y)]^{\gamma} \approx 0$, which might be the case at younger ages, especially if $\gamma \approx 0.2$ or some other small value, then

$$
\bar{\rho}(x,y) \approx \rho(x,y)[\bar{s}_c(x,y)]^{\gamma}\,. \qquad (12^*)
$$

Equation (1*) implies

$$
\frac{\bar{\rho}(x,y)}{\rho(x,y)} = [\bar{s}_c(x,y)]^{\gamma}\,. \qquad (13)
$$

For instance, if $\bar{s}_c(x,y) = 0.01$ and $\gamma = 0.2$, then $\frac{\bar{\rho}(x,y)}{\rho(x,y)} = 0.4$, i.e. the observed rate of progress in reducing population death rates is only 40% of the rate of progress on the individual level. More generally, because survivorship is less than one and γ is positive, eq. (1*) implies that the age-specific pace of progress in reducing mortality over time is always less for a population than for the individuals who comprise the population. This is an important theoretical conclusion. We hope that the equations presented in this Letter will also lead to empirical results about the values of $\rho(x,y)$.

References

Vaupel, J.W. and T.I. Missov. 2014. "Unobserved population heterogeneity: A review of formal relationships." *Demographic Research* 31:659–688.