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Research Article

Analyzing hyperstable population models

Robert Schoen

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Analyzing hyperstable population models

Robert Schoen¹

Abstract

OBJECTIVE

Few methods are available for analyzing populations with changing rates. Here hyperstable models are presented and substantially extended to facilitate such analyses.

METHODS

Hyperstable models, where a known birth trajectory yields a consistent set of age-specific birth rates, are set out in both discrete and continuous form. Mathematical analysis is used to find new relationships between model functions for a range of birth trajectories.

RESULTS

Hyperstable population projection matrices can create bridges that project any given initial population to any given ending population. New, explicit relationships are found between period and cohort births for exponential, polynomial, and sinusoidal birth trajectories. In quadratic and cubic models, the number of cohort births equals the number of period births a generation later, with a modest adjustment. In sinusoidal models, cohort births equal the number of period births a generation later, modified by a factor related to cycle length.

CONTRIBUTION

Because of their adaptability, structure, and internal relationships, hyperstable birth models afford a valuable platform for analyzing populations with changing fertility. The new relationships found provide insight into dynamic models and period-cohort connections and offer useful applications to analysts.

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1. Introduction

Hyperstable birth models expand the stable population concept by allowing births to vary freely over time, and they offer a valuable framework for demographic analysis. The flexibility in their structure also leads to useful relationships and applications, especially with regard to cyclical models and to period-cohort translations.

1.1 Introducing the hyperstable birth model

Hyperstable models are derived from a known birth trajectory – that is, a time series of total population births – and yield a consistent set of underlying age-specific birth rates. To begin with the model’s discrete formulation, let \mathbf{G}_t be an $N \times N$ diagonal matrix whose j^{th} diagonal element is g_{t+1-j} , where g_t is the number of births (persons in the first age group) at time t . Let \mathbf{F} be a time-invariant $N \times N$ Leslie-form matrix whose subdiagonal elements are all 1. Thus the model ignores mortality below the highest age of reproduction: age group N . Eliminating mortality simplifies the description of the model with little loss, as mortality effects can be added later. Half of the births are assumed female and half male.

The first-row elements, f_j , are the fertility values or, more precisely, give the number of persons in the first age group at the end of the interval per person in the j th age group at the beginning of the interval. The f_j are scaled to sum to 1. Now let \mathbf{g}_t be an $N \times 1$ -column vector whose j th element is g_{jt} , the number of persons in the j th age group at time t . In the absence of mortality, those persons are the births g_{t+1-j} . These definitions yield the basic hyperstable projection relationship

$$\mathbf{g}_t = \mathbf{G}_t \mathbf{F} \mathbf{G}_{t-1}^{-1} \mathbf{g}_{t-1} \quad (1)$$

The hyperstable population projection matrix (PPM) that takes the population from time $t-1$ to time t , \mathbf{A}_t , is then given by

$$\mathbf{A}_t = \mathbf{G}_t \mathbf{F} \mathbf{G}_{t-1}^{-1} \quad (2)$$

Note that diagonal matrices \mathbf{G}_t and \mathbf{G}_{t-1} have a substantial overlap in their elements. The j th diagonal element ($1 \leq j < N$) of \mathbf{G}_{t-1} is the $(j+1)$ st diagonal element of \mathbf{G}_t . With the elements of \mathbf{G} and \mathbf{F} known, Equation (2) gives the age-specific fertility rates at every time point. Each fertility rate in \mathbf{A}_t has the form $f_j g_t / g_{t-j}$ and thus represents a fixed fertility fraction adjusted by a ratio of current to past births.

An illustrative example may be helpful. With $N = 3$ and $f_1 + f_2 + f_3 = 1$, we can write

$$\mathbf{G}_t = \begin{bmatrix} g_t & 0 & 0 \\ 0 & g_{t-1} & 0 \\ 0 & 0 & g_{t-2} \end{bmatrix} \quad (3)$$

$$\mathbf{F} = \begin{bmatrix} f_1 & f_2 & f_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (4)$$

$$\mathbf{A}_t = \begin{bmatrix} f_1 g_t / g_{t-1} & f_2 g_t / g_{t-2} & f_3 g_t / g_{t-3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (5)$$

When \mathbf{A}_t multiplies \mathbf{g}_{t-1} , the (1,1) element of the product is g_t , the (2,1) element is g_{t-1} , and the (3,1) element is g_{t-2} , accomplishing the projection. The age-specific fertility values in \mathbf{A}_t are the constant fertility fractions f_j adjusted by a ratio of beginning and ending populations (current and past births). The fraction of births arising from the j th age group is always simply f_j .

The hyperstable model is also expressible in continuous form. The (known) number of births at each exact time t , $g(t)$, leads to age-specific fertility rates of the form

$$\phi(x,t) = [f(x) g(t) / g(t-x)] \quad (6)$$

where $f(x)$ is the constant fraction of births at exact age x . Summing those rates over all reproductive ages gives the total fertility rate (TFR) at time t , $\text{TFR}(t)$, as

$$\text{TFR}(t) = \int f(x) g(t) / [.5 g(t-x)] dx \quad (7)$$

as half of the population is female.

1.2 Background literature

The hyperstable model was introduced in Schoen and Kim (1994), which described its basic features and overall dynamics. Considering a range of birth trajectories, Schoen and Kim (1994) showed that hyperstable models had quite reasonable patterns of age-specific birth rates.

Other dynamic models – that is, models with changing vital rates – have been examined over the past half century. The pioneering work of Coale (1972) examined various patterns of fertility change and their implications for growth, while Lee (1974) considered the implications of some constrained patterns of fertility. Kim (1987) found a general algebraic expression linking fertility rates to births, but in most cases it is too complex to yield closed form equations. Schoen and Kim (1994), Kim and Schoen (1996), and especially Schoen and Jonsson (2003) explored the metastable model, where fertility rates that changed exponentially over age and time produced quadratic exponential growth. The metastable model stands out, as it provides a closed form relationship for the birth sequence generated by a specified dynamic fertility sequence, though it is restricted to monotonic changes in fertility rates over time.

Cyclical models have been of special interest. Skellam (1967) introduced cyclically stable populations, which were further examined in Namboodiri (1969), Coale (1972), Tuljapurkar (1985, 1990), and Schoen and Kim (1994). Going beyond fixed vital rates, Coale (1972: Chapter 6) and Tuljapurkar (1985) used Fourier series to relate sinusoidal fertility to birth trajectories. While those two analyses were advances, both needed to assume small oscillations and consider only the first harmonic of the Fourier series, and neither presented closed form relationships between sinusoidal fertility trajectories and their resultant birth trajectories.

Going from fertility rates to births, the usual approach in past work, has the advantage of going from behavior to its consequences. Still, going from births to an underlying schedule of fertility rates is also of analytical value. Doing so, Kim and Schoen (1996) examined a sinusoidal birth function with constant proportional fertility rates and showed it yielded a closed form expression for the TFR trajectory when the age-specific rates followed a normal curve. That work provided a detailed examination, over a wide range of cycle lengths, of the relationship between births and TFRs, showing phase differences and relative amplifications. Because fertility rates were proportional over age, that model was not hyperstable.

In a distinct line of fertility research, analysts have examined the connections between period and cohort fertility. The initial “translation” paper by Ryder (1964) developed a linear relationship between the cohort total fertility rate (CFR) for the cohort born at time t and the period TFR a generation later. That approach was modified and elaborated by Keilman (2001), Bongaarts and Feeney (1998), and Zeng and Land (2002). An alternative approach, assuming constant proportional age-specific fertility, was

presented in Schoen (2022). That analysis showed that even when the period TFR followed a cubic trajectory, the TFR values reproduced the CFR for the cohort born a generation earlier, with a modest adjustment. Up to now, period-cohort relationships have not been examined in the context of hyperstable models.

The present objective is to consider hyperstable models with a broad range of birth trajectories to present new and potentially useful relationships in those models and to indicate some possible applications. Distinctive aspects of the mathematical structure of hyperstable birth models are discussed, new period-cohort relationships are developed, and further regularities in cyclical models are explored. We begin with a closer examination of the basic model.

2. The basic hyperstable birth model

The basic model of Equations (1) through (6) produces a set of age-specific birth rates consistent with any given birth trajectory g . The age pattern of those birth rates has been found to be reasonable over a range of birth trajectories (Schoen and Kim 1994). Under hyperstability, there must be a fixed proportion of births by age of mother, hence a constant period mean age of childbearing.

2.1 The mathematical structure of hyperstable models

Over time, the product of the projection matrices, \mathbf{A}_t , simplifies. Consider the product of successive PPMs that project the population from time $t-1$ to time $t+m$. From Equation (2), that product matrix, $\mathbf{M}_{t-1,m}$, is given by

$$\begin{aligned} \mathbf{M}_{t-1,m} &= \mathbf{A}_{t+m} \mathbf{A}_{t+m-1} \dots \mathbf{A}_t \\ &= (\mathbf{G}_{t+m} \mathbf{F} \mathbf{G}_{t+m-1}^{-1}) (\mathbf{G}_{t+m-1} \mathbf{F} \mathbf{G}_{t+m-2}^{-1}) \dots (\mathbf{G}_{t+1} \mathbf{F} \mathbf{G}_t^{-1}) (\mathbf{G}_t \mathbf{F} \mathbf{G}_{t-1}^{-1}) \\ &= \mathbf{G}_{t+m} \mathbf{F}^{(m+1)} \mathbf{G}_{t-1}^{-1} \end{aligned} \quad (8)$$

As successive terms “telescope,” Leslie-form matrix \mathbf{F} is raised to increasingly higher powers over time, and the latent or eigenstructure of \mathbf{F} becomes manifest. That eigenstructure has a fixed and relatively simple structure, which is described in Appendix

A. When \mathbf{F} can be expressed solely in terms of its constant dominant left and right eigenvectors, the model can be seen as having converged to hyperstability.

2.2 Hyperstable projection matrices as bridges

The telescoping of projection matrices shown in Equation (8) has an important implication. Each PPM can serve as a bridge from an interval's initial population to its ending population. Under hyperstability, each PPM's rates are adjusted so that the composition of the initial population is accommodated to yield the ending population (see Schoen 2006: 153–154). Perturbations in birth trajectories are immediately incorporated into the fertility rates because there is a fixed proportion of births by age of mother. Barring an extreme fluctuation in the number of births, the age pattern of fertility remains plausible.

Analysts seeking to find a PPM that takes a given initial population to a given ending population can thus use a hyperstable form projection matrix, with f_j values appropriate to the circumstances. More generally, if the f_j can be chosen by the investigator, any such PPM can be written in hyperstable form. To show that, assume that the g_t trajectory is known. Let 3×3 PPM \mathbf{A}_t take the population from time $t-1$ to time t , with the first row elements of \mathbf{A}_t being any known values a , b , and c . From the projection relationship,

$$g_t = a g_{t-1} + b g_{t-2} + c g_{t-3} \quad (9)$$

To put PPM \mathbf{A}_t in hyperstable form with $f_1 + f_2 + f_3 = 1$, we can write

$$a = f_1 g_t / g_{t-1} ; b = f_2 g_t / g_{t-2} ; c = f_3 g_t / g_{t-3}$$

or, rearranging,

$$f_1 = a g_{t-1} / g_t ; f_2 = b g_{t-2} / g_t ; f_3 = c g_{t-3} / g_t \quad (10)$$

From Equations (9) and (10), we can verify that the sum of the f values is 1; Equation (2) then applies. An analytical example can be seen in the discussion after Equations (3)–(5), and an application to a stable/stationary transition is provided in Section 6.1. The approach in Equations (9) and (10) holds for PPMs of any size and demonstrates that hyperstable relationships are implicit in population projections.

3. Hyperstable models with exponential birth trajectories

Birth trajectory $g(t)$ is often exponential in analytical population models. Here we consider three different types of exponential trajectories.

3.1 Models with simple exponential birth trajectories

Let $g(t)$ be the simple exponential e^{rt} , the growth rate that is characteristic of stable populations. From Equation (6), the fertility rate at age x and time t , $\phi(x,t)$, has the form

$$\phi(x,t) = f(x) g(t) / g(t-x) = f(x) e^{rt} / e^{r(t-x)} = f(x) e^{rx} \quad (11)$$

Consistent with fixed rate stable population assumptions, hyperstable age-specific fertility is constant over time. In other words, the stable population is a hyperstable population with a simple exponential birth trajectory.

3.2 Models with quadratic and higher exponential birth trajectories

Let the exponentially quadratic hyperstable birth trajectory be

$$g(t) = \exp[st + ht^2] \quad (12)$$

Then, simplifying the $g(t)/g(t-x)$ ratio,

$$\phi(x,t) = f(x) \exp[sx + h(2xt - x^2)] \quad (13)$$

The quadratic exponential hyperstable birth trajectory is the same as the metastable birth trajectory, but the two models have different age-specific birth rates (Schoen and Jonsson 2003).

The approach used in Equation (13) can produce age-specific birth rates for exponentially cubic and higher-order hyperstable birth trajectories, though such higher-order models are not frequently seen in demographic work.

3.3 Models with declining exponential birth trajectories

Hyperstable models can capture an exponential decline in fertility to stationarity. For $b > 0$, let

$$g(t) = a + d \exp[-bt] \quad (14)$$

When $a \neq 0$, the age-specific rates vary over time. Though different schedules of birth rates can produce that birth sequence, we focus on hyperstable values. Then the age-specific birth rates can be found as above, though those expressions do not simplify.

In the hyperstable scenario, new relationships can be found between period and cohort birth trajectories. Indeed, hyperstable models lend themselves to cohort relationships. Specifically, the number of births to the cohort born in year τ can be written as

$$C(\tau) = \int [f(x) g(\tau+x)] dx \quad (15)$$

Unless stated otherwise, integrals range over all reproductive ages, here taken to be ages 15 to 45. The cohort TFR is then given by $CFR(\tau) = C(\tau) / [.5 g(\tau)]$.

To evaluate the integral in Equation (15), we need to specify age-specific fertility density function $f(x)$. A simple but convenient and quite reasonable assumption is the quadratic function

$$f(x) = -3/20 + x/75 - x^2/4500 = (x-15)(45-x)/4500 \quad (16)$$

The derivation of Equation (16) is shown in Appendix B. The value of $f(x)$ is 0 at ages 15 and 45 and reaches a maximum of 0.05 at age 30. The area under the symmetric $f(x)$ curve is 1, with mean 30, variance 45, standard deviation 6.71, and 0 skew.

Using Equation (16), Equations (14) and (15) give

$$C(\tau) = \int (-3/20 + x/75 - x^2/4500) (a + d \exp[-b(\tau+x)]) dx \quad (17)$$

The integration in Equation (17) can be performed explicitly, as shown in Appendix C. The result is

$$C(\tau) = a + d \exp[-b(\tau+30)] J(b) \quad (18)$$

where moderating factor $J(b)$ is solely a function of rate of decline b and is given by

$$J(b) = \{ \exp[15b] (15b-1) + \exp[-15b] (15b+1) \} / (2250 b^3) \quad (19)$$

Factor $J(b)$ varies nonlinearly with rate of decline b and can become unstable when $b < 0.00001$. When b is in the more realistic range from 0.001 to 0.5, $J(b)$ increases approximately exponentially with b (see Table 1), going from about 1.000 to 41.786.

Table 1: Period and cohort births and values of moderating factor $J(b)$, hyperstable models with exponentially declining birth trajectories

Rate of decline (b)	Factor $J(b)$	Cohort births ($C(0)$)	Period births ($g(30)$)
0.001	.9999	1.9703	1.9704
0.0025	1.0001	1.9279	1.9277
0.005	1.0006	1.8612	1.8607
0.01	1.0023	1.7425	1.7408
0.025	1.0141	1.4790	1.4724
0.05	1.0574	1.2359	1.2231
0.1	1.2439	1.0619	1.0498
0.25	3.3293	1.0018	1.0006
0.5	41.7859	1.0000	1.0000

Note: Calculations are based on Equations (14), (18), and (19). Parameters are $a = 1$ and $d = 1$. At time 0, $g(0) = a+d = 2$. As the rate of decline increases, $J(b)$ increases and the number of births goes to $a = 1$.

Equation (18) indicates that the number of cohort births at time τ lags period births by a generation (30 years), with the declining exponential term in $g(\tau+30)$ moderated by $J(b)$. For $0.001 < b < 0.045$, adjustment factor $J(b)$ is small, less than 1.05. The results in Table 1, for a wide range of rates of fertility decline, show that cohort births closely approximate the number of period births 30 years later.

4. Hyperstable models with polynomial birth trajectories

Here we examine linear, quadratic, and cubic birth trajectories, focusing on the connection between the period $g(t)$ and cohort $C(\tau)$ birth functions. The evaluation of the integrals involved is shown in Appendix D.

4.1 Hyperstable models with linear birth trajectories

Let the birth trajectory have the linear form

$$g(t) = R + at \quad (20)$$

Then the cohort born in year τ has total births numbering

$$C(\tau) = \int [R+a(\tau+x)] f(x) dx \quad (21)$$

where $f(x)$ is the quadratic density function of Equation (16). Carrying out the integration yields

$$C(\tau) = [R + a(\tau+30)] = g(\tau+30) \quad (22)$$

The cohort born in year τ has exactly the number of births over its lifetime as occur during year $\tau+30$ – that is, the births of a generation later.

4.2 Hyperstable models with quadratic birth trajectories

Let the birth trajectory be

$$g(t) = R + at + bt^2 \quad (23)$$

With $C(\tau) = \int [R+a(\tau+x)+b(\tau+x)^2] f(x) dx$ and $f(x)$ as in Equation (16), the number of cohort births is

$$C(\tau) = g(\tau+30) + 45b \quad (24)$$

Equation (24) shows that with a quadratic birth trajectory, the number of cohort births is the number of period births 30 years later plus the product of the variance of $f(x)$ times quadratic parameter b . That additive adjustment is constant over time and reflects the interaction between the nonlinearity of the birth sequence and the variability in the underlying birth density distribution. Here, as elsewhere, once the period births to cohort births relationship is established, it can be used to go from cohort births to period births. Thus if the known $C(\tau)$ is given by Equation (24) or the cohort TFR is given by $C(\tau)/g(\tau)$, then the birth sequence is given by Equation (23).

4.3 Hyperstable models with cubic birth trajectories

Let the birth trajectory be

$$g(t) = R + at + bt^2 + dt^3 \quad (25)$$

With $C(\tau) = \int [R+a(\tau+x)+b(\tau+x)^2+d(\tau+x)^3] f(x) dx$ and $f(x)$ given by Equation (16), the number of births to the cohort is

$$C(\tau) = g(\tau+30) + 45 [b + 3d (\tau+30)] \tag{26}$$

With a cubic birth trajectory, the number of cohort births is the number of period births 30 years later plus additive adjustments for the variance of $f(x)$ and the quadratic and cubic parameters (see Appendix D). Note that if $d = 0$ in Equation (25), Equation (23) arises, and if $b = 0$ in Equation (23), Equation (21) arises. With all three polynomials, the results parallel the relationships between period and cohort TFRs found by Schoen (2022).

In sum, under linear, quadratic, and cubic birth trajectories, the number of births to a cohort is the number of period births a generation later with adjustments for the nonlinearity of the birth trajectory. For most populations, the quadratic and cubic parameters are small, so the additive adjustment is modest.

5. The cyclical hyperstable model

Cyclical models have long been of interest to demographers. Under cyclical hyperstability, the birth trajectory repeats itself every K intervals, and in each successive cycle, the number of births at every stage of the cycle is λ times the number at that stage in the preceding cycle. The fertility values in each PPM of every cycle follow from the hyperstable ($f_j g_t / g_{t-j}$) relationships of Equation (6). Hence the population at every stage and the rate of growth from cycle to cycle follow from the specified cyclically stable birth trajectory, with the PPMs determined accordingly.

The flexibility of the hyperstable model allows any within-cycle birth trajectory and inter-cycle rate of growth. Within a cycle, the births can be skewed, follow a sawtooth pattern, or take on any specified shape. The fixed fraction of births by age of mother is still determined by the f_j , and age-specific birth rates at every stage in a cycle are repeated at that stage in the next cycle.

5.1 The cycle population projection matrix

The cycle PPM (or CPM) is the product of K consecutive PPMs that take the population from a given stage in one cycle to the same stage in the next cycle. Hence CPM_j is the product of the $(j+1)$ st PPM in a cycle through the following j th PPM of the next cycle. For example, CPM_1 is the product $(\lambda \mathbf{A}_1 \mathbf{A}_K \mathbf{A}_{K-1} \dots \mathbf{A}_3 \mathbf{A}_2)$ and projects the population at

the first stage in one cycle to the first stage of the next. Given the telescoping nature of the PPMs, we can use Equations (2) and (8) to write

$$\text{CPM}_1 = \lambda \mathbf{G}_1 \mathbf{F}^K \mathbf{G}_1^{-1} \quad (27)$$

Analogous relationships exist for every stage in the cycle. Equation (27) shows that the CPM has Sykes form (Sykes 1973), meaning that it is the product of (1) a constant parameter, (2) a diagonal matrix, (3) a row stochastic matrix, and (4) the inverse of the diagonal matrix. In Sykes form, λ is the dominant eigenvalue of CPM_1 , and the diagonal of \mathbf{G}_1 contains the dominant right eigenvector of CPM_1 . While Tuljapurkar (1985: 4) noted that a CPM is generally a complicated object that resists analysis, in cyclical hyperstable models, the dominant eigenstructure is quite accessible.

5.2 Relationships under cyclical hyperstationarity

When $\lambda = 1$, the model is cyclically hyperstationary and does not grow from cycle to cycle. Cyclical hyperstationary models have not previously been examined and contain a number of noteworthy relationships, including those between period and cohort birth trajectories. More broadly, cyclically stationary models can be useful analytically as they provide a framework for examining long-term zero-growth populations that have short-term fluctuations in fertility and are thus more flexible than stationary population models.

Within a cycle, the birth trajectory can take any shape. Here we focus on the sinusoidal hyperstationary model

$$g(t) = a + b \sin \omega t \quad (28)$$

where cycle frequency $\omega = 2\pi/T$ and T is the length of a complete cycle. The number of births to the cohort born in year τ can then be written

$$C(\tau) = \int [a + b \sin \omega(\tau+x)] f(x) dx \quad (29)$$

Carrying out the integration with $f(x)$ as in Equation (16), as shown in Appendix E, yields the result

$$C(\tau) = a + b \sin \omega(\tau+30) H(\omega) \quad (30)$$

where moderating factor $H(\omega)$ is given by

$$H(\omega) = [\sin 15\omega - 15\omega \cos 15\omega] / (1125 \omega^3) \quad (31)$$

Equations (30) and (31) are new and indicate that when the birth trajectory varies sinusoidally, cohort births also vary sinusoidally with the same cycle length. There is a constant phase difference of 30 years (the mean of the $f(x)$) between the period and cohort birth curves. The relative amplitude of those curves around the base level set by parameter a is determined by factors b and $H(\omega)$. Thus the phase difference and relative amplitude are separable, with the relative amplitude depending only on parameter b and the cycle length.

Table 2 examines the nature and effects of moderating factor $H(\omega)$. While observed population cycles typically are some 30 to 60 years long, with American fertility having cycles of around 40 years, Table 2 casts a broader net and shows values of $H(\omega)$, $g(30)$ and $C(0)$, and $g(40)$ and $C(10)$ for cycle lengths from 15 to 1,000 years. Some short cycle lengths have negative $H(\omega)$ values, with $H(\omega)$ equal to 0 when $T = 20.9747$ years. At that cycle length, period births vary over time, but the number of cohort births remains constant. As shown in Figure 1, the value of $H(\omega)$ then increases steadily with T and approaches 1 for large T . While period and cohort births fluctuate with the same cycle length, $C(\tau)$ is always closer to base value $a = 2$ than is $g(\tau+30)$.

Table 2: Period and cohort births and values of moderating factor $H(\omega)$, hyperstationary models with sinusoidal birth trajectories

Cycle length (T)	Factor $H(\omega)$	Period births $g(30)$	Cohort births $C(0)$	Period births $g(40)$	Cohort births $C(10)$
15	-0.0760	2.0000	2.0000	1.6536	2.0263
20	-0.0287	2.0000	2.0000	2.0000	2.0000
20.9747	0.0000	2.1696	2.0000	1.7795	2.0000
24	0.1025	2.4000	2.0410	1.6536	1.9645
30	0.3040	2.0000	2.0000	2.3464	2.1053
40	0.5443	1.6000	1.7823	2.0000	2.0000
50	0.6869	1.7649	1.8385	1.6196	1.7387
60	0.7740	2.0000	2.0000	1.6536	1.7319
70	0.8301	2.1736	2.1441	1.8264	1.8559
80	0.8679	2.2828	2.2455	2.0000	2.0000
90	0.8945	2.3464	2.3099	2.1368	2.1224
100	0.9139	2.3804	2.3479	2.2351	2.2149
200	0.9780	2.3236	2.3165	2.3804	2.3720
1000	0.9991	2.0750	2.0749	2.0995	2.0994

Note: Period birth trajectory given by $g(t) = a + b \sin \omega t$, with $T = 2\pi/\omega$, $a = 2$, and $b = 0.4$.

Figure 1: Moderating factor $H(\omega)$ in cyclically stationary hyperstable models, by cycle length T from 5 to 100 years

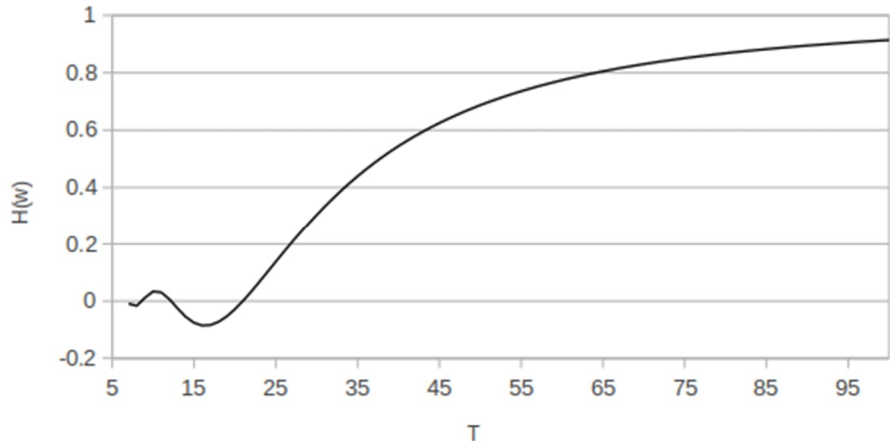


Figure 2 shows the time trajectories of births for periods, $g(t+30)$, and cohorts, $C(t)$. Cycle lengths of 20, 40, 60, and 80 years are shown, with $a = 2$ and $b = 0.4$. The $g(t+30)$ curve always goes from a peak of 2.4 births to a trough of 1.6 births. The curves for $g(t+30)$ and $C(t)$ are always in phase, with crossovers when $\sin\omega(t+30)$ is 0 – that is, midway between the minimum and maximum values of $g(t+30)$. When $T = 20$, crossovers occur at time 10 and multiples of 10 ($T/2$), and the fluctuations in $C(t)$ are small and opposite in direction to those of $g(t+30)$. For longer cycle lengths, the $C(t)$ curve comes progressively closer to the $g(t+30)$ curve.

To confirm Equations (30) and (31), calculations were done using hypothetical data with $T = 24$, $a = 2$, and $b = 0.3$. It follows that $H = 0.1025$. Single years of age and time were used. Fertility follows the quadratic in Equation (16), but $f(x)$ is evaluated at mid-interval – that is, at ages 15.5 through 44.5. The array of births was then calculated for every age and for times 0 through 52. The 53-year time span was needed, as the cohorts born in years (-15) through $(+8)$ had to be followed through age 44. The total number of cohort births for each year was found by summing the diagonals of the age-specific birth array. That yielded data for 24 periods and 24 cohorts, a complete cycle. Details of the calculation, and the elements of the age-time birth array, are provided in Appendix F.

Figure 2: Time trajectories of period (time $t+30$) and cohort (born time t) births in sinusoidal hyperstationary models where $a = 2$ and $b = 0.4$, with cycle lengths of 20, 40, 60, and 80 years

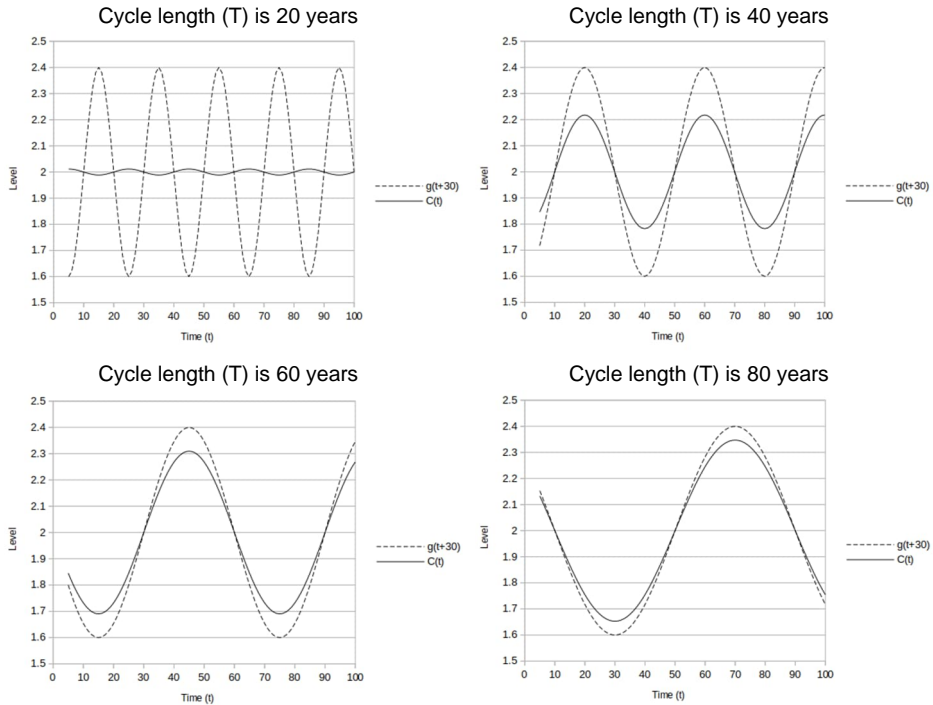


Table 3 compares the number of cohort births found from Equations (30) and (31) with those calculated from the age-time birth array. Discrepancies are to be expected when comparing the results of a continuous-time equation with those of a discrete-time calculation, but here they are quite small. The Equation (30) cohort birth figures track the discrete figures very closely. The average error is about one-quarter of a percent (0.00256), and the magnitude of the largest error is 0.004. Table 3 thus confirms the relationship between period and cohort births given by Equations (30) and (31).

Table 3: Numerical comparison of cohort births calculated from Equation (30) with births calculated from a birth array based on hypothetical data

(1) Cohort year of birth (t)	(2) Period births $g(t+15)$	(3) Cohort births from Eq(30)	(4) Cohort births from age-time birth array	(5) Proportional error $[(3)-(4)]/(3)$
-15	100	97.83	98.14	-0.0032
-14	107.76	97.34	97.57	-0.0024
-13	115	97.03	97.17	-0.0015
-12	121.21	96.92	96.96	-0.0004
-11	125.98	97.03	96.96	0.0007
-10	128.98	97.34	97.17	0.0017
-9	130	97.83	97.57	0.0026
-8	128.98	98.46	98.14	0.0033
-7	125.98	99.20	98.93	0.0038
-6	121.21	100	99.60	0.0040
-5	115	100.80	100.40	0.0039
-4	107.76	101.54	101.17	0.0036
-3	100	102.17	101.86	0.0030
-2	92.24	102.66	102.43	0.0023
-1	85	102.97	102.83	0.0014
0	78.79	103.08	103.04	0.0004
1	74.02	102.97	103.04	-0.0006
2	71.02	102.66	102.83	-0.0016
3	70	102.17	102.43	-0.0025
4	71.02	101.54	101.86	-0.0032
5	74.02	100.80	101.17	-0.0037
6	78.79	100	100.40	-0.0040
7	85	99.20	99.60	-0.0040
8	92.24	98.46	98.83	-0.0037
9	100	97.83	98.14	-0.0032

Notes: Birth trajectory $g(t) = a + b \sin \omega t$. Equation (30) parameters are $a=100$, $b=30$, and $T=2\pi/\omega=24$. Factor $H(\omega) = 0.1025$. Average absolute error over a cycle is 0.00256.
 Source: See text and Appendix F.

5.3 Some relevant model comparisons

The present period-cohort birth results can be compared with the findings of Kim and Schoen (1996), which analyzed the same sinusoidal $g(t)$ function but assumed constant, normal curve-based fertility proportions by age and focused on the relationship between the birth and period net reproduction (NRR) curves. Kim and Schoen (1996) found that the $g(t)$ and NRR curves had the same cycle length but that the NRR curve was not

sinusoidal. Moreover, both the relative amplitude and phase difference between the curves varied nonlinearly with the ratio of the cycle length to the mean age of net maternity. Thus between the $g(t)$ and NRR curves, Kim and Schoen (1996) did not find the simple relationship that characterizes hyperstable period and cohort birth trajectories.

The period–cohort birth relationship for polynomials has a direct parallel in the relationship between period and cohort TFRs found in Schoen (2022). Extending that earlier approach to sine curves, we assume constant proportional age-specific fertility over time and denote the time t TFR by $R(t)$. Then the TFR for the cohort born in year τ can be written

$$CFR(\tau) = \int R(\tau+x) f(x) dx \tag{32}$$

where the integral ranges over ages 15 to 45 and $f(x)$ is the quadratic density of Equation (16). With

$$R(t) = a + b \sin \omega t \tag{33}$$

the integral in Equation (32) has the same form as the integral in Equation (29). Integrating, per Appendix E, yields the new result

$$CFR(\tau) = a + b \sin \omega (\tau+30) H(\omega) \tag{34}$$

As in Equation (30), $H(\omega)$ is given by Equation (31). Thus, with sinusoidal change, constant proportional fertility gives a simple relationship between TFRs and CFRs, while hyperstability gives a simple relationship between the number of period and cohort births.

6. Applications of the hyperstable model

Core features of hyperstable models are the connection between births and birth rates, the fixed proportion of births by age of mother, and the relationship between the number of period and cohort births. These features lead to a number of useful applications, three of which are presented in this section.

6.1 Transitioning from a stable to a stationary population

Hyperstable models can specify transition matrices that bridge from one fertility regime to another. Consider a stable population growing at rate r . For simplicity and without loss

of generality, assume there are only three reproductive age groups and that at time t we want to transition to a stationary population by fixing the number of births (persons in the first age group) at the time t level – that is, e^{rt} . Then the hyperstable projection matrix that takes the population from time t to time $t+1$ can be written using Equations (1) and (2), with PPM \mathbf{A}_{t+1} given by

$$\mathbf{A}_{t+1} = \begin{bmatrix} f_1 & f_2 e^r & f_3 e^{2r} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (35)$$

where the j th exponent in the first row reflects the g_{t+1}/g_{t+1-j} ratio with $g_{t+1} = g_t$. The time $t+1$ population, \mathbf{g}_{t+1} , thus has elements e^{rt} , e^{rt} , and $e^{r(t-1)}$. Projecting over the next time interval with first-row elements f_1 , f_2 , and $f_3 e^r$ yields a time $t+2$ population with e^{rt} persons in each age group. That stationary population can be further projected using the \mathbf{F} matrix of Equation (4), and this approach can be used with any set of fertility values that sum to 1.

Schoen (2020) advanced the Constant- α approach to bridge multistate populations across an interval by pre- and post-multiplying by diagonal matrices. That approach preserved the cross-product ratios of the base matrix and thus provided a maximum likelihood solution. In a Leslie matrix, all cross-product ratios are 0, so it is not meaningful to speak of preserving such ratios. However, hyperstability and Constant- α are linked, as both transform a base matrix using pre- and post-multiplication by diagonal matrices to take a given beginning-of-interval population to a specified end-of-interval population. The Constant- α approach can thus be seen as hyperstability applied to a multistate context.

6.2 Estimating cohort proportions childless from numbers of births

The relationships previously developed make it possible to estimate cohort ultimate parity distributions from birth sequences. In this subsection, we show how to do so by estimating cohort proportions childless.

Previous sections provided equations for translating numbers of period births to numbers of cohort births when the period birth sequence is a declining exponential [Eq(17)]; a linear, quadratic, or cubic polynomial [Eq(26)]; or a sinusoidal curve

[Eq(30)]. Given cohort births at time t , $C(t)$, the TFR for the cohort born at time t , $CFR(t)$, is simply

$$CFR(t) = C(t) / [.5 g(t)] \quad (36)$$

Using the conventional, though implicit, assumption that age-specific fertility rates are the same for women at all parities, the ultimate (here age 45) parity distribution follows a Poisson distribution determined by the level of fertility (Krishnamoorthy 1979; see Tuljapurkar et al. 2020 for an extended analysis). The number at ultimate parity j for the cohort born year t is thus

$$Par(j) = \exp[-CFR(t)] CFR(t)^j / j! \quad (37)$$

When $j = 0$, the proportion childless is simply

$$Par(0) = \exp[-CFR(t)] \quad (38)$$

With a CFR of 2, $Par(0)$ is 0.135, indicating that just under 1 in 7 women have no children. Using Equation (37), the complete parity distribution for a birth cohort can readily be determined for any of the birth sequences examined.

6.3 The variability of cohort fertility in cyclically stationary hyperstable models

We can now consider how the level of cohort fertility varies by location (or stage) within each cycle of a cyclically stationary hyperstable model. Specifically, with the CFR given by Equation (36), we can examine how the size of a cohort is related to the level of cohort fertility and see whether large cohorts have higher or lower fertility than small cohorts.

To find the maximum and minimum number of period births, let birth sequence $g(t)$ be the sine function in Equation (28). Differentiating $g(t)$ and setting the derivative equal to 0 gives

$$g'(t) = b \cos \omega t / \omega = 0 \quad \text{or simply} \quad \cos \omega t = \cos 2\pi t/T = 0 \quad (39)$$

The cosine is 0 at $\pi/2$ radians and at $3\pi/2$ radians, so the extrema for births, t_{ext} , are when

$$\pi/2 \text{ or } 3\pi/2 = 2\pi t_{ext} / T$$

which occur at times

$$t_{\text{ext}} = T/4 \text{ and } 3T/4 \quad (40)$$

Equation (28) then yields the maximum and minimum number of births.

With cohort total fertility given by Equation (36), Equations (28), (30), and (31) yield

$$\text{CFR}(t) = \{ a + b \sin \omega(t+30) [\sin 15\omega - 15\omega \cos 15\omega] / (1125\omega^3) \} / \{ .5[a + b \sin \omega t] \} \quad (41)$$

Differentiating Equation (41), setting the derivative equal to 0 and rearranging, as shown in Appendix G, yields the numerically tractable relationship

$$aH(\omega) \cos \omega(t+30) - ab \cos \omega t = bH(\omega) \sin 30\omega \quad (42)$$

where $H(\omega)$ is given by Equation (31). For any given frequency or cycle length, Equation (42) allows the cycle stage of the minimum and maximum values to be found, and from them the minimum and maximum CFR values.

Figure 3 plots the $g(t)$ and $\text{CFR}(t)$ functions for cycle lengths 20, 40, 60, and 80 years. While the $g(t)$ curve is a simple sine wave, the $\text{CFR}(t)$ curve of Equation (41) is not. It is asymmetrical with respect to variations around mean birth value 2, with respect to the midpoint of the cycle, and with respect to the $g(t)$ curve. Nonetheless, there is a strong tendency for large birth cohorts to have small CFRs and vice versa.

Table 4 quantifies those impressions, showing the times of the birth and fertility maxima and minima and their associated birth and CFR values. For cycle lengths 20 and 60, the maximum CFR aligns perfectly with the minimum g , and the minimum CFR with the maximum g . For cycle lengths 40 and 80, the alignment is off by 7%–9%. For all cycle lengths examined, the variability of the CFR exceeds that of the number of births, which always ranges between 1.7 and 2.3. The greatest CFR variation, when $T = 60$ years, is 1.09 children, which is quite substantial for a stationary population.

Figure 3: Time trajectories of period births $[g(t)]$ and cohort total fertility for cohorts born at time t $[CFR(t)]$ in sinusoidal hyperstationary models where $a = 2$ and $b = 0.3$, with cycle lengths of 20, 40, 60, and 80 years

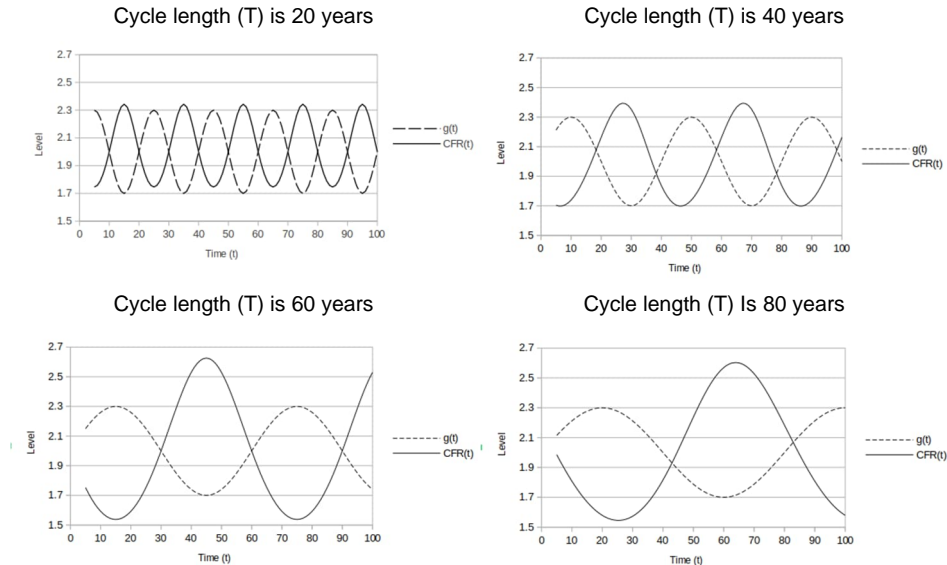


Table 4: Extrema of birth sequence $g(t)$ and fertility level $CFR(t)$ in cyclically stationary hyperstable models, with cycle lengths of 20, 40, 60, and 80 years

Cycle length (T)	g(t)		CFR(t)		g(t)/CFR(t)				
	Max t	Min t	Max value	Min value	max t	min t			
20	5	15	2.3	1.7	15	5	2.343	1.747	0
40	10	30	2.3	1.7	27.284	6.370	2.395	1.698	.091
60	15	45	2.3	1.7	45	15	2.626	1.537	0
80	20	60	2.3	1.7	63.948	25.306	2.603	1.545	-.066

Note: See text. Parameter values are $a = 2$ and $b = 0.3$. The $g(t)/CFR(t)$ phase deviation is defined as $[g(\max t) - CFR(\min t)] / T$.

Table 5 looks at the implications of within-cycle fertility level variations by relating the maximum and minimum CFR values to their associated proportions childless. There is considerable variation in childlessness by cycle stage and some by cycle length. The largest swing in the proportion childless is when $T = 60$, where the minimum is 7.2%

childless and the maximum is 21.5%, a 14.3% difference. Even in a long-term stationary population, cohort experience can vary substantially within a cycle.

Table 5: Maximum and minimum proportions childless in cyclically stationary hyperstable model cohorts, with cycle lengths of 20, 40, 60, and 80 years

Cycle length (T)	Proportion childless at		Difference (min – max)
	Max CFR	Min CFR	
20	0.096	0.174	0.078
40	0.091	0.183	0.092
60	0.072	0.215	0.143
80	0.074	0.213	0.139

Note: Age-specific fertility assumed the same for women of all parities.

7. Summary and conclusions

Hyperstable models typically begin with a known birth trajectory and generate a consistent set of age-specific birth rates. The model’s underlying mathematical structure can be stated explicitly and leads to a number of new relationships and applications. Hyperstable population projection matrices can bridge, or project a population across, an interval, going from any given initial population to any given ending population. That bridging property means that the fertility elements of any PPM can be written in hyperstable form – that is, as $[f_x g_t / g_{t-x}]$.

Hyperstable models allow the derivation of new closed form equations relating cohort birth trajectories to period birth trajectories. Such relationships are derived for exponential, polynomial, and cyclical birth trajectories. Cyclical models are examined further, with a new closed form expression found for the phase and relative amplitude differences between period and cohort birth curves. The new theoretical relationship between cohort and period birth curves in Equation (30) was examined using hypothetical data and confirmed numerically. In all trajectories examined, a close relationship was found between the total number of births to a given year’s cohort and the number of period births a generation later.

Hyperstable models are a valuable resource for the analysis of populations with time-varying fertility. The models offer an adaptable and realistic framework for analyzing birth trajectories and show the close connections between period and cohort births. Several model applications are presented, showing how hyperstable models can bridge between fertility regimes, find cohort fertility levels and proportions childless from birth sequences, and illuminate intra-cycle variability in cyclical populations. Due

to their flexibility, hyperstable models can provide a useful analytical platform that captures the past and likely future of actual populations more fully than can fixed rate stationary or stable models.

8. Acknowledgments

Valuable comments from Lowell Hargens are acknowledged with thanks. Appreciation is also due to Lowell Hargens for producing Figures 1 through 3. This paper is dedicated to the memory of Young J. Kim, my collaborator on past work with hyperstable models.

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Appendix A. The eigenstructure of Leslie-form Matrix \mathbf{F}

Eigenstructure refers to the unique decomposition of PPMs that breaks down an $N \times N$ PPM \mathbf{A} into

$$\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{-1} \tag{A.1}$$

where \mathbf{U} is an $N \times N$ matrix of right eigenvectors and $\mathbf{\Lambda}$ is an $N \times N$ diagonal matrix of eigenvalues. The (1,1) element of $\mathbf{\Lambda}$ is the dominant eigenvalue (or root) of \mathbf{A} , often denoted by λ . It gives the long-term (or stable) growth rate implied by \mathbf{A} . The leftmost column of \mathbf{U} is the dominant right eigenvector, frequently denoted by \mathbf{u} . It gives the long-term (stable) population composition implied by \mathbf{A} . Matrix \mathbf{U}^{-1} , sometimes denoted by \mathbf{V} , is the matrix of left eigenvectors. The first row of \mathbf{V} , often represented by \mathbf{v}' , is the dominant left eigenvector and relates to reproductive values (cf. Schoen 2006).

The eigenstructure of row stochastic Leslie-form matrices (projection matrices whose rows sum to 1), such as the 3×3 \mathbf{F} matrix of Equation (4), is relatively simple. For all N , the dominant root (λ) is 1. For $N = 3$, the right eigenvector matrix, \mathbf{U}_F , has the form

$$\mathbf{U}_F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/\lambda_2 & 1/\lambda_3 \\ 1 & 1/\lambda_2^2 & 1/\lambda_3^2 \end{bmatrix} \tag{A.2}$$

with λ_j denoting the j th subordinate eigenvalue, which must have a magnitude of less than 1. The generalization to $N > 3$ is clear. The eigenvalue matrix, $\mathbf{\Lambda}$, is a diagonal matrix whose (1,1) element is 1 and whose j th diagonal element ($j > 1$) is λ_j . The dominant right eigenvector of \mathbf{F} , \mathbf{u}_F , is the first column of \mathbf{U}_F —that is, a column vector of 1s. The left eigenvector matrix, \mathbf{V}_F , is the inverse of \mathbf{U}_F . The dominant left eigenvector of \mathbf{F} , denoted by \mathbf{v}_F' , has elements that sum to 1. Specifically, the j th element of \mathbf{v}_F' , v_{jF} , can be written

$$v_{jF} = \frac{\sum_{i=j}^N f_i}{\sum_{i=1}^N \sum_{j=1}^N f_i} \tag{A.3}$$

When $N = 3$, the elements of \mathbf{v}_F' are $1/D$, $(f_2+f_3)/D$, and f_3/D , where $D = 1 + f_2 + 2f_3$.

Sykes form, from Sykes (1973), is an alternative eigenstructure-related decomposition. In Sykes form, PPM \mathbf{A} can be written

$$\mathbf{A} = \lambda \mathbf{U}_d \mathbf{F} \mathbf{U}_d^{-1} \quad (\text{A.4})$$

where \mathbf{U}_d is a diagonal matrix whose diagonal elements are the elements of the dominant right eigenvector of \mathbf{A} and where \mathbf{F} is a row stochastic matrix. Any PPM can be written, uniquely, in Sykes form.

Appendix B. Derivation of the quadratic $f(x)$ curve in Equation (16)

The essential characteristics of base fertility curve $f(x)$ can be captured reasonably well by a simple parabola. With $f(x)$ equal to 0 at ages 15 and 45, undefined outside that interval, and having an area of 1 in that interval, we can write the specifying equations

$$\begin{aligned} 0 &= w + 15y + 15^2 z \\ 0 &= w + 45y + 45^2 z \\ 1 &= \int (w + yx + zx^2) dx = wx + yx^2/2 + zx^3/3 \end{aligned} \quad (\text{B.1})$$

where the integral in the last equation is evaluated for x between ages 15 and 45. Solving the three equations in Equation (B.1) gives the solution

$$w = -3/20, \quad y = 1/75, \quad \text{and } z = -1/4500 \quad (\text{B.2})$$

The resultant function, $f(x) = -3/20 + x/75 - x^2/4500$, is symmetric at about age 30 and reaches a maximum value of 0.05 at that age.

Appendix C. Derivation of Equation (18) (declining exponential trajectory)

The declining exponential $g(t)$ function in Equation (14) leads to the integral in Equation (15) for the trajectory of cohort births:

$$C(\tau) = \int [f(x) g(\tau+x)] dx \quad (C.1)$$

where $f(x)$ is the quadratic function of Equation (16) and the integral is evaluated between ages 15 and 45. Thus

$$\begin{aligned} C(\tau) &= \int [a+de^{-b(\tau+x)}] [-3/20 + x/75 - x^2/4500] dx \\ &= a + de^{-b\tau} \{ -(3/20)[-e^{-bx}/b]_{15}^{45} + (1/75) [-e^{-bx}(1+bx)/b^2]_{15}^{45} \\ &\quad - (1/4500) [-x^2 e^{-bx}/b - (2/b)(e^{-bx}/b^2 + x e^{-bx}/b)]_{15}^{45} \} \\ &= a + (de^{-b\tau}/4500b^3) \{ e^{-45b}(30b+2) + e^{-15b}(30b-2) \} \\ &= a + (de^{-b(\tau+30)}) \{ [e^{15b}(15b-1) + e^{-15b}(150b+1)] / 2250b^3 \} \end{aligned} \quad (C.2)$$

which are the results shown in Equations (18) and (19). The integrated expression, $[...]_{15}^{45}$, is to be evaluated between ages 15 and 45.

That derivation uses the known integral relationships

$$\int x e^{ax} dx = e^{ax} (ax - 1) / a^2$$

and

$$\int x^2 e^{ax} dx = x^2 e^{ax} / a - (2/a) \int x e^{ax} dx \quad (C.3)$$

(Beyer 1978: 368).

Appendix D. Derivation of Equation (26) (cubic trajectory)

Since the cohort birth trajectory that results from a period cubic birth trajectory reduces to the linear and quadratic relationships of Equations (22) and (24), only the cubic case is shown here. The following derivation is analogous to the derivation in online Appendix C to Schoen (2022).

With birth trajectory

$$g(t) = R + at + bt^2 + dt^3 \tag{D.1}$$

and the quadratic $f(x)$ of Equation (16), we can write

$$\begin{aligned} C(\tau) &= \int [R + a(\tau+x) + b(\tau+x)^2 + d(\tau+x)^3] [-3/20 + x/75 - x^2/4500] dx \\ &= g(\tau) + [a+2b\tau+3d\tau^2] \int [(-3x/20)+(x^2/75)-(x^3/4500)]dx \\ &\quad + [b+3d\tau] \int [(-3x^2/20)+(x^3/75)-(x^4/4500)]dx \\ &\quad + d \int [(-3x^3/20)+(x^4/75)-(x^5/4500)]dx \end{aligned}$$

Integrating and evaluating over the 15 to 45 age range, we have

$$\begin{aligned} C(\tau) &= g(\tau) + 30(a+2b\tau+3d\tau^2) + 945(b+3d\tau) + 31050d \\ &= R + a(\tau+30) + b(\tau+30)^2 + 45b + d(\tau+30)^3 + 135d\tau + 4050d \\ &= g(\tau+30) + 45 [b + 3d(\tau+30)] \end{aligned} \tag{D.2}$$

which is the result in Equation (26).

Appendix E. Derivation of Equation (30) (sinusoidal trajectory)

With birth trajectory

$$g(t) = a + b \sin \omega t \tag{E.1}$$

where frequency ω is related to cycle length T by $\omega = 2\pi/T$ and $f(x)$ is the quadratic of Equation (16), the number of births to the cohort born in year τ can be written

$$\begin{aligned} C(\tau) &= \int [a + b \sin \omega(\tau+x)] [-3/20 + x/75 - x^2/4500] dx \tag{E.2} \\ &= a + b \int [\sin \omega\tau \cos \omega x + \cos \omega\tau \sin \omega x] [-3/20 + x/75 - x^2/4500] dx \\ &= a+b \int [\sin \omega\tau \cos \omega x] [-3/20+x/75-x^2/4500]dx \\ &\quad + \int [\cos \omega\tau \sin \omega x] [-3/20+x/75-x^2/4500]dx \end{aligned}$$

where the integrals range from age 15 to age 45 and use is made of the trigonometric identity $\sin(y+z) = \sin y \cos z + \cos y \sin z$ (Beyer 1978: 168). Let the first integral in the last equation in (E.2) be denoted by Y and the second integral by Z . Then, carrying out the integration using the expressions for $(\int x^j \sin ax dx)$ and $(\int x^j \cos ax dx)$ in Beyer 1978 (350, 358), we find

$$Y = [2 \sin 45\omega - 2 \sin 15\omega - 30\omega (\cos 45\omega + \cos 15\omega)] / (4500\omega^3)$$

and

$$Z = [-2 \cos 45\omega + 2 \cos 15\omega - 30\omega (\sin 45\omega + \sin 15\omega)] / (4500\omega^3) \quad (\text{E.3})$$

Using Equation (E.3) in Equation (E.2) and rearranging, we have

$$\begin{aligned} C(\tau) = a + [b/(2250\omega^3)] & [\sin \omega\tau \sin 45\omega - \sin \omega\tau \sin 15\omega \\ & - 15\omega(\sin \omega\tau \cos 45\omega + \sin \omega\tau \cos 15\omega) - \cos \omega\tau \cos 45\omega \\ & + \cos \omega\tau \cos 15\omega - 15\omega (\cos \omega\tau \sin 45\omega + \cos \omega\tau \sin 15\omega)] \end{aligned} \quad (\text{E.4})$$

Using trigonometric angle sum relationships like the one following Equation (E.2) (see Beyer 1978: 167), we can rewrite Equation (E.4) as

$$\begin{aligned} C(\tau) = a + (b/(2250\omega^3)) & [-\cos \omega(\tau+45) + \cos \omega(\tau+15) \\ & - 15\omega (\sin \omega(\tau+45) + \sin \omega(\tau+15))] \end{aligned}$$

and using function sum trigonometric relationships (Beyer 1978: 168), we then arrive at

$$C(\tau) = a + b \sin \omega(\tau+30) [(\sin 15\omega - 15\omega \cos 15\omega) / (1125\omega^3)] \quad (\text{E.5})$$

which reproduces Equations (30) and (31). A similar derivation can express cohort total fertility in terms of sinusoidal TFR(t) parameters under the assumptions used in Schoen (2022).

Appendix F. Calculations to confirm the Equation (30) relationship between period and cohort births, using hypothetical data

Equations (30) and (31) present a new relationship between period and cohort births under hyperstability, showing a fixed phase difference between those two trajectories and a relative amplitude in their fluctuations that greatly depends on cycle length. The correctness of that relationship is not obvious, and numerical confirmation is appropriate.

Assume that the birth trajectory follows the sinusoidal pattern of Equation (28) and that the fertility density is described by Equation (16). Set cycle length T to 24 years and use parameters $a = 2$ and $b = 0.3$. Using single years of age, we adjust fertility to mid-year and let $f_{x+1/2}$ be

$$f_{x+1/2} = -3/20 + (x+1/2) / 75 - (x+1/2)^2 / 4500 \quad (\text{F.1})$$

The values of $f_{x+\frac{1}{2}}$ are shown in the second column of Table F-1.

Table F-1: Comparing calculated and estimated births in hypothetical data

$f(x)$ from	Eq(16)	Time(t)	0	1	2	3	4	5	
Discrete		g(t)	100	107.7646	115	121.2132	125.9808	128.9778	
		C(t-15)	97.82565	97.33697	97.02978	96.925	97.02978	97.33697	
Mid-year	$f(x+.5)$	Adjusted f	Age	Array of age-specific births					
Age	Eq(15)	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.327596	0.353037	0.37674	0.397095	0.412713	0.422531
16.5	0.0095	0.0094947	16	0.949472	1.023225	1.091925	1.15092	1.196187	1.224644
17.5	0.0152778	0.0152693	17	1.526929	1.645457	1.755935	1.850805	1.9236	1.969362
18.5	0.0206111	0.0205997	18	2.059966	2.21995	2.369	2.496992	2.595204	2.656942
19.5	0.0255	0.0254858	19	2.548583	2.746488	2.93089	3.08924	3.210746	3.287128
20.5	0.0299444	0.0299278	20	2.99278	3.225178	3.44172	3.627669	3.770353	3.860047
21.5	0.0339444	0.0339256	21	3.392558	3.656021	3.90149	4.11228	4.274024	4.3757
22.5	0.0375	0.0374792	22	3.747916	4.038909	4.310085	4.54295	4.721633	4.833958
23.5	0.0406111	0.0405885	23	4.058854	4.374056	4.667735	4.919923	5.113434	5.235079
24.5	0.0432778	0.0432537	24	4.325373	4.661249	4.974211	5.242957	5.449172	5.578805
25.5	0.0455	0.0454747	25	4.547472	4.900594	5.229626	5.512171	5.728976	5.865265
26.5	0.0472778	0.0472515	26	4.725151	5.092092	5.433981	5.727567	5.952844	6.094458
27.5	0.0486111	0.0485841	27	4.85841	5.235634	5.587161	5.889023	6.12065	6.266257
28.5	0.0495	0.0494725	28	4.947249	5.331329	5.689281	5.99666	6.232521	6.380789
29.5	0.0499444	0.0499167	29	4.991669	5.379284	5.740456	6.0506	6.288582	6.438184
30.5	0.0499444	0.0499167	30	4.991669	5.379284	5.740456	6.0506	6.288582	6.438184
31.5	0.0495	0.0494725	31	4.947249	5.331329	5.689281	5.99666	6.232521	6.380789
32.5	0.0486111	0.0485841	32	4.85841	5.235634	5.587161	5.889023	6.12065	6.266257
33.5	0.0472778	0.0472515	33	4.725151	5.092092	5.433981	5.727567	5.952844	6.094458
34.5	0.0455	0.0454747	34	4.547472	4.900594	5.229626	5.512171	5.728976	5.865265
35.5	0.0432778	0.0432537	35	4.325373	4.661249	4.974211	5.242957	5.449172	5.578805
36.5	0.0406111	0.0405885	36	4.058854	4.374056	4.667735	4.919923	5.113434	5.235079
37.5	0.0375	0.0374792	37	3.747916	4.038909	4.310085	4.54295	4.721633	4.833958
38.5	0.0339444	0.0339256	38	3.392558	3.656021	3.90149	4.11228	4.274024	4.3757
39.5	0.0299444	0.0299278	39	2.99278	3.225178	3.44172	3.627669	3.770353	3.860047
40.5	0.0255	0.0254858	40	2.548583	2.746488	2.93089	3.08924	3.210746	3.287128
41.5	0.0206111	0.0205997	41	2.059966	2.21995	2.369	2.496992	2.595204	2.656942
42.5	0.0152778	0.0152693	42	1.526929	1.645457	1.755935	1.850805	1.9236	1.969362
43.5	0.0095	0.0094947	43	0.949472	1.023225	1.091925	1.15092	1.196187	1.224644
44.5	0.0032778	0.003276	44	0.327596	0.353037	0.37674	0.397095	0.412713	0.422531
	1.0005556	0.9999996	g(t) check	99.99996	107.765	115.0005	121.2137	125.9813	128.9783
			Array C	98.13621	97.57096	97.17126	96.96436	96.96436	97.17127
			% error	-0.00317	-0.0024	-0.00146	-0.00041	0.000674	0.001702
			ABS % err	0.003175	0.002404	0.001458	0.000406	0.000674	0.001702

Table F-1: (Continued)

$f(x)$ from	$Eq(16)$	Time(t)	6	7	8	9	10	11	
Discrete		$g(t)$	130	128.9778	125.9807	121.2132	114.9999	107.7645	
		$C(t-15)$	97.82565	98.4625	99.20413	100	100.7959	101.5375	
Midyear	$f(x+.5)$	Adjusted f							
Age	$Eq(15)$	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.42588	0.422531	0.412713	0.397094	0.37674	0.353037
16.5	0.0095	0.0094947	16	1.23435	1.224644	1.196187	1.150919	1.091924	1.023224
17.5	0.0152778	0.0152693	17	1.98497	1.969361	1.9236	1.850804	1.755934	1.645456
18.5	0.0206111	0.0205997	18	2.678	2.656942	2.595203	2.496991	2.368999	2.219949
19.5	0.0255	0.0254858	19	3.31318	3.287127	3.210745	3.089239	2.930889	2.746486
20.5	0.0299444	0.0299278	20	3.89064	3.860046	3.770351	3.627667	3.441718	3.225176
21.5	0.0339444	0.0339256	21	4.41038	4.3757	4.274022	4.112278	3.901488	3.656018
22.5	0.0375	0.0374792	22	4.87227	4.833958	4.721632	4.542948	4.310083	4.038906
23.5	0.0406111	0.0405885	23	5.27657	5.235078	5.113432	4.919921	4.667733	4.374053
24.5	0.0432778	0.0432537	24	5.62302	5.578804	5.449171	5.242954	4.974207	4.661246
25.5	0.0455	0.0454747	25	5.91175	5.865264	5.728974	5.512168	5.229622	4.900591
26.5	0.0472778	0.0472515	26	6.14276	6.094457	5.952842	5.727564	5.433977	5.092088
27.5	0.0486111	0.0485841	27	6.31592	6.266256	6.120648	5.88902	5.587157	5.23563
28.5	0.0495	0.0494725	28	6.43136	6.380788	6.232519	5.996657	5.689277	5.331325
29.5	0.0499444	0.0499167	29	6.48921	6.438183	6.28858	6.050597	5.740452	5.379281
30.5	0.0499444	0.0499167	30	6.48921	6.438183	6.28858	6.050597	5.740452	5.379281
31.5	0.0495	0.0494725	31	6.43136	6.380788	6.232519	5.996657	5.689277	5.331325
32.5	0.0486111	0.0485841	32	6.31592	6.266256	6.120648	5.88902	5.587157	5.23563
33.5	0.0472778	0.0472515	33	6.14276	6.094457	5.952842	5.727564	5.433977	5.092088
34.5	0.0455	0.0454747	34	5.91175	5.865264	5.728974	5.512168	5.229622	4.900591
35.5	0.0432778	0.0432537	35	5.62302	5.578804	5.449171	5.242954	4.974207	4.661246
36.5	0.0406111	0.0405885	36	5.27657	5.235078	5.113432	4.919921	4.667733	4.374053
37.5	0.0375	0.0374792	37	4.87227	4.833958	4.721632	4.542948	4.310083	4.038906
38.5	0.0339444	0.0339256	38	4.41038	4.3757	4.274022	4.112278	3.901488	3.656018
39.5	0.0299444	0.0299278	39	3.89064	3.860046	3.770351	3.627667	3.441718	3.225176
40.5	0.0255	0.0254858	40	3.31318	3.287127	3.210745	3.089239	2.930889	2.746486
41.5	0.0206111	0.0205997	41	2.678	2.656942	2.595203	2.496991	2.368999	2.219949
42.5	0.0152778	0.0152693	42	1.98497	1.969361	1.9236	1.850804	1.755934	1.645456
43.5	0.0095	0.0094947	43	1.23435	1.224644	1.196187	1.150919	1.091924	1.023224
44.5	0.0032778	0.003276	44	0.42588	0.422531	0.412713	0.397094	0.37674	0.353037
	1.0005556	0.9999996	$g(t)$ check	130.0005	128.9783	125.9812	121.2136	115.0004	107.7649
			Array C	97.57097	98.13624	98.82855	99.60071	100.4001	101.1723
			% error	0.002603	0.003314	0.003786	0.003993	0.003926	0.003597
			ABS % err	0.002603	0.003314	0.003786	0.003993	0.003926	0.003597

Table F-1: (Continued)

$f(x)$ from	Eq(16)	Time(t)	12	13	14	15	16	17	
Discrete		g(t)	99.99992	92.23534	84.99992	78.78672	74.01918	71.0222	
		C(t-15)	102.1744	102.663	102.9702	103.075	102.9702	102.663	
Midyear	f(x+5)	Adjusted f	Age	Array of age-specific births					
Age	Eq(15)	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.3276	0.302163	0.27846	0.258105	0.242487	0.232669
16.5	0.0095	0.0094947	16	0.949499	0.875775	0.807074	0.74808	0.702812	0.674356
17.5	0.0152778	0.0152693	17	1.526899	1.408341	1.297864	1.202995	1.130199	1.084438
18.5	0.0206111	0.0205997	18	2.059998	1.900048	1.750998	1.623007	1.524795	1.463057
19.5	0.0255	0.0254858	19	2.548598	2.35071	2.166308	2.007958	1.886453	1.810072
20.5	0.0299444	0.0299278	20	2.992798	2.760419	2.543878	2.357929	2.215246	2.125552
21.5	0.0339444	0.0339256	21	3.392597	3.129176	2.883707	2.672918	2.511175	2.409499
22.5	0.0375	0.0374792	22	3.747897	3.456888	3.185712	2.952848	2.774165	2.661841
23.5	0.0406111	0.0405885	23	4.058897	3.74374	3.450062	3.197874	3.004365	2.88272
24.5	0.0432778	0.0432537	24	4.325396	3.989548	3.676586	3.407841	3.201626	3.071994
25.5	0.0455	0.0454747	25	4.547496	4.194402	3.865371	3.582826	3.366022	3.229734
26.5	0.0472778	0.0472515	26	4.725196	4.358304	4.016416	3.72283	3.497554	3.355941
27.5	0.0486111	0.0485841	27	4.858396	4.481162	4.129636	3.827774	3.596148	3.450542
28.5	0.0495	0.0494725	28	4.947196	4.563067	4.205116	3.897737	3.661877	3.51361
29.5	0.0499444	0.0499167	29	4.991696	4.604112	4.242941	3.932797	3.694816	3.545215
30.5	0.0499444	0.0499167	30	4.991696	4.604112	4.242941	3.932797	3.694816	3.545215
31.5	0.0495	0.0494725	31	4.947196	4.563067	4.205116	3.897737	3.661877	3.51361
32.5	0.0486111	0.0485841	32	4.858396	4.481162	4.129636	3.827774	3.596148	3.450542
33.5	0.0472778	0.0472515	33	4.725196	4.358304	4.016416	3.72283	3.497554	3.355941
34.5	0.0455	0.0454747	34	4.547496	4.194402	3.865371	3.582826	3.366022	3.229734
35.5	0.0432778	0.0432537	35	4.325396	3.989548	3.676586	3.407841	3.201626	3.071994
36.5	0.0406111	0.0405885	36	4.058897	3.74374	3.450062	3.197874	3.004365	2.88272
37.5	0.0375	0.0374792	37	3.747897	3.456888	3.185712	2.952848	2.774165	2.661841
38.5	0.0339444	0.0339256	38	3.392597	3.129176	2.883707	2.672918	2.511175	2.409499
39.5	0.0299444	0.0299278	39	2.992798	2.760419	2.543878	2.357929	2.215246	2.125552
40.5	0.0255	0.0254858	40	2.548598	2.35071	2.166308	2.007958	1.886453	1.810072
41.5	0.0206111	0.0205997	41	2.059998	1.900048	1.750998	1.623007	1.524795	1.463057
42.5	0.0152778	0.0152693	42	1.526899	1.408341	1.297864	1.202995	1.130199	1.084438
43.5	0.0095	0.0094947	43	0.949499	0.875775	0.807074	0.74808	0.702812	0.674356
44.5	0.0032778	0.003276	44	0.3276	0.302163	0.27846	0.258105	0.242487	0.232669
	1.0005556	0.9999996	g(t) check	100.0003	92.23571	85.00026	78.78704	74.01948	71.02248
			Array C	101.8646	102.4298	102.8295	103.0364	103.0364	102.8295
			% error	0.003032	0.002271	0.001366	0.000374	-0.00064	-0.00162
			ABS % err	0.003032	0.002271	0.001366	0.000374	0.000643	0.001622

Table F-1: (Continued)

$f(x)$ from	Eq(16)	Time(t)	18	19	20	21	22	23	
		g(t)	70	71.02226	74.01931	78.7869	85.00013	92.23558	
Discrete		C(t-15)	102.1744	101.5375	100.7959	100	99.20413	98.4625	
Midyear	f(x+.5)	Adjusted f	Age	Array of age-specific births					
Age	Eq(15)	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.22932	0.232669	0.242487	0.258106	0.27846	0.302164
16.5	0.0095	0.0094947	16	0.66465	0.674356	0.702813	0.748082	0.807076	0.875777
17.5	0.0152778	0.0152693	17	1.06883	1.084439	1.130201	1.202997	1.297867	1.408345
18.5	0.0206111	0.0205997	18	1.442	1.463059	1.524798	1.62301	1.751003	1.900053
19.5	0.0255	0.0254858	19	1.78402	1.810073	1.886456	2.007963	2.166313	2.350716
20.5	0.0299444	0.0299278	20	2.09496	2.125554	2.21525	2.357934	2.543884	2.760426
21.5	0.0339444	0.0339256	21	2.37482	2.409501	2.511179	2.672924	2.883714	3.129184
22.5	0.0375	0.0374792	22	2.62353	2.661843	2.77417	2.952854	3.18572	3.456897
23.5	0.0406111	0.0405885	23	2.84123	2.882722	3.00437	3.197881	3.45007	3.74375
24.5	0.0432778	0.0432537	24	3.02778	3.071997	3.201631	3.407848	3.676596	3.989558
25.5	0.0455	0.0454747	25	3.18325	3.229737	3.366028	3.582834	3.865381	4.194413
26.5	0.0472778	0.0472515	26	3.30764	3.35944	3.49756	3.722838	4.016426	4.358316
27.5	0.0486111	0.0485841	27	3.40088	3.450545	3.596154	3.827783	4.129646	4.481173
28.5	0.0495	0.0494725	28	3.46304	3.513613	3.661883	3.897745	4.205126	4.563079
29.5	0.0499444	0.0499167	29	3.49419	3.545218	3.694822	3.932806	4.242951	4.604123
30.5	0.0499444	0.0499167	30	3.49419	3.545218	3.694822	3.932806	4.242951	4.604123
31.5	0.0495	0.0494725	31	3.46304	3.513613	3.661883	3.897745	4.205126	4.563079
32.5	0.0486111	0.0485841	32	3.40088	3.450545	3.596154	3.827783	4.129646	4.481173
33.5	0.0472778	0.0472515	33	3.30764	3.35944	3.49756	3.722838	4.016426	4.358316
34.5	0.0455	0.0454747	34	3.18325	3.229737	3.366028	3.582834	3.865381	4.194413
35.5	0.0432778	0.0432537	35	3.02778	3.071997	3.201631	3.407848	3.676596	3.989558
36.5	0.0406111	0.0405885	36	2.84123	2.882722	3.00437	3.197881	3.45007	3.74375
37.5	0.0375	0.0374792	37	2.62353	2.661843	2.77417	2.952854	3.18572	3.456897
38.5	0.0339444	0.0339256	38	2.37482	2.409501	2.511179	2.672924	2.883714	3.129184
39.5	0.0299444	0.0299278	39	2.09496	2.125554	2.21525	2.357934	2.543884	2.760426
40.5	0.0255	0.0254858	40	1.78402	1.810073	1.886456	2.007963	2.166313	2.350716
41.5	0.0206111	0.0205997	41	1.442	1.463059	1.524798	1.62301	1.751003	1.900053
42.5	0.0152778	0.0152693	42	1.06883	1.084439	1.130201	1.202997	1.297867	1.408345
43.5	0.0095	0.0094947	43	0.66465	0.674356	0.702813	0.748082	0.807076	0.875777
44.5	0.0032778	0.003276	44	0.22932	0.232669	0.242487	0.258106	0.27846	0.302164
	1.0005556	0.9999996	g(t) check	70.00028	71.02254	74.0196	78.78721	85.00047	92.23595
			Array C	102.4298	101.8646	101.1722	100.4001	99.60067	98.82851
			% error	-0.0025	-0.00322	-0.00373	-0.004	-0.004	-0.00372
			ABS % err	0.0025	0.003221	0.003734	0.004001	0.003997	0.003717

Table F-1: (Continued)

$f(x)$ from	Eq(16)	Time(t)	24	25	26	27	28	29	
Discrete		g(t)	100.0002	107.7647	115.0002	121.2133	125.9809	128.9778	
		C(t-15)	97.82565	97.33697	97.02978	96.925	97.02978	97.33697	
Midyear	f(x+.5)	Adjusted f	Age	Array of age-specific births					
Age	Eq(15)	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.327601	0.353037	0.37674	0.397095	0.412713	0.422531
16.5	0.0095	0.0094947	16	0.949502	1.023226	1.091926	1.150921	1.196188	1.224644
17.5	0.0152778	0.0152693	17	1.526902	1.64546	1.755937	1.850806	1.923602	1.969362
18.5	0.0206111	0.0205997	18	2.060003	2.219954	2.369003	2.496995	2.595206	2.656943
19.5	0.0255	0.0254858	19	2.548604	2.746492	2.930894	3.089243	3.210748	3.287129
20.5	0.0299444	0.0299278	20	2.992805	3.225183	3.441725	3.627673	3.770355	3.860048
21.5	0.0339444	0.0339256	21	3.392606	3.656026	3.901495	4.112284	4.274027	4.375702
22.5	0.0375	0.0374792	22	3.747906	4.038914	4.310091	4.542954	4.721637	4.83396
23.5	0.0406111	0.0405885	23	4.058907	4.374063	4.667741	4.919928	5.113437	5.235081
24.5	0.0432778	0.0432537	24	4.325407	4.661256	4.974217	5.242961	5.449176	5.578807
25.5	0.0455	0.0454747	25	4.547507	4.900601	5.229632	5.512176	5.728979	5.865267
26.5	0.0472778	0.0472515	26	4.725208	5.092099	5.433987	5.727572	5.952847	6.09446
27.5	0.0486111	0.0485841	27	4.858408	5.235642	5.587167	5.889029	6.120654	6.266259
28.5	0.0495	0.0494725	28	4.947208	5.331337	5.689288	5.996666	6.232525	6.380791
29.5	0.0499444	0.0499167	29	4.991708	5.379292	5.740463	6.050606	6.288586	6.438186
30.5	0.0499444	0.0499167	30	4.991708	5.379292	5.740463	6.050606	6.288586	6.438186
31.5	0.0495	0.0494725	31	4.947208	5.331337	5.689288	5.996666	6.232525	6.380791
32.5	0.0486111	0.0485841	32	4.858408	5.235642	5.587167	5.889029	6.120654	6.266259
33.5	0.0472778	0.0472515	33	4.725208	5.092099	5.433987	5.727572	5.952847	6.09446
34.5	0.0455	0.0454747	34	4.547507	4.900601	5.229632	5.512176	5.728979	5.865267
35.5	0.0432778	0.0432537	35	4.325407	4.661256	4.974217	5.242961	5.449176	5.578807
36.5	0.0406111	0.0405885	36	4.058907	4.374063	4.667741	4.919928	5.113437	5.235081
37.5	0.0375	0.0374792	37	3.747906	4.038914	4.310091	4.542954	4.721637	4.83396
38.5	0.0339444	0.0339256	38	3.392606	3.656026	3.901495	4.112284	4.274027	4.375702
39.5	0.0299444	0.0299278	39	2.992805	3.225183	3.441725	3.627673	3.770355	3.860048
40.5	0.0255	0.0254858	40	2.548604	2.746492	2.930894	3.089243	3.210748	3.287129
41.5	0.0206111	0.0205997	41	2.060003	2.219954	2.369003	2.496995	2.595206	2.656943
42.5	0.0152778	0.0152693	42	1.526902	1.64546	1.755937	1.850806	1.923602	1.969362
43.5	0.0095	0.0094947	43	0.949502	1.023226	1.091926	1.150921	1.196188	1.224644
44.5	0.0032778	0.003276	44	0.327601	0.353037	0.37674	0.397095	0.412713	0.422531
	1.0005556	0.9999996	g(t) check	100.0006	107.7652	115.0006	121.2138	125.9814	128.9783
			Array C						
			% error	Ave % err					
			ABS % err	0.002563					

Table F-1: (Continued)

$f(x)$ from	$Eq(16)$	Time(t)	30	31	32	33	34	35	
		$g(t)$	130	128.9777	125.9807	121.213	114.9998	107.7643	
Discrete		$C(t-15)$	97.82565	98.4625	99.20413	100	100.7959	101.5375	
Midyear	$f(x+.5)$	Adjusted f							
Age	$Eq(15)$	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.42588	0.422531	0.412713	0.397094	0.376739	0.353036
16.5	0.0095	0.0094947	16	1.23435	1.224643	1.196186	1.150918	1.091923	1.023222
17.5	0.0152778	0.0152693	17	1.98497	1.969361	1.923599	1.850802	1.755932	1.645454
18.5	0.0206111	0.0205997	18	2.678	2.656941	2.595201	2.496989	2.368996	2.219945
19.5	0.0255	0.0254858	19	3.31318	3.287126	3.210743	3.089236	2.930885	2.746482
20.5	0.0299444	0.0299278	20	3.89064	3.860045	3.770349	3.627664	3.441714	3.225171
21.5	0.0339444	0.0339256	21	4.41038	4.375698	4.27402	4.112274	3.901483	3.656013
22.5	0.0375	0.0374792	22	4.87227	4.833956	4.721629	4.542944	4.310078	4.0389
23.5	0.0406111	0.0405885	23	5.27657	5.235077	5.113429	4.919916	4.667727	4.374047
24.5	0.0432778	0.0432537	24	5.62302	5.578802	5.449167	5.242949	4.974201	4.661239
25.5	0.0455	0.0454747	25	5.91175	5.865262	5.72897	5.512163	5.229616	4.900583
26.5	0.0472778	0.0472515	26	6.14276	6.094455	5.952838	5.727559	5.433971	5.092081
27.5	0.0486111	0.0485841	27	6.31592	6.266254	6.120644	5.889015	5.58715	5.235623
28.5	0.0495	0.0494725	28	6.43136	6.380786	6.232515	5.996652	5.68927	5.331318
29.5	0.0499444	0.0499167	29	6.48921	6.438181	6.288576	6.050592	5.740445	5.379273
30.5	0.0499444	0.0499167	30	6.48921	6.438181	6.288576	6.050592	5.740445	5.379273
31.5	0.0495	0.0494725	31	6.43136	6.380786	6.232515	5.996652	5.68927	5.331318
32.5	0.0486111	0.0485841	32	6.31592	6.266254	6.120644	5.889015	5.58715	5.235623
33.5	0.0472778	0.0472515	33	6.14276	6.094455	5.952838	5.727559	5.433971	5.092081
34.5	0.0455	0.0454747	34	5.91175	5.865262	5.72897	5.512163	5.229616	4.900583
35.5	0.0432778	0.0432537	35	5.62302	5.578802	5.449167	5.242949	4.974201	4.661239
36.5	0.0406111	0.0405885	36	5.27657	5.235077	5.113429	4.919916	4.667727	4.374047
37.5	0.0375	0.0374792	37	4.87227	4.833956	4.721629	4.542944	4.310078	4.0389
38.5	0.0339444	0.0339256	38	4.41038	4.375698	4.27402	4.112274	3.901483	3.656013
39.5	0.0299444	0.0299278	39	3.89064	3.860045	3.770349	3.627664	3.441714	3.225171
40.5	0.0255	0.0254858	40	3.31318	3.287126	3.210743	3.089236	2.930885	2.746482
41.5	0.0206111	0.0205997	41	2.678	2.656941	2.595201	2.496989	2.368996	2.219945
42.5	0.0152778	0.0152693	42	1.98497	1.969361	1.923599	1.850802	1.755932	1.645454
43.5	0.0095	0.0094947	43	1.23435	1.224643	1.196186	1.150918	1.091923	1.023222
44.5	0.0032778	0.003276	44	0.42588	0.422531	0.412713	0.397094	0.376739	0.353036
	1.0005556	0.9999996	$g(t)$ check	130.0005	128.9782	125.9812	121.2135	115.0003	107.7648

Table F-1: (Continued)

$f(x)$ from	$Eq(16)$	Time(t)	36	37	38	39	40	41	
		$g(t)$	99.99976	92.23519	84.99978	78.78661	74.0191	71.02215	
Discrete		$C(t-15)$	102.1744	102.663	102.9702	103.075	102.9702	102.663	
Midyear	$f(x+5)$	Adjusted f	Age	Array of age-specific births					
Age	$Eq(15)$	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.327599	0.302162	0.278459	0.258105	0.242487	0.232669
16.5	0.0095	0.0094947	16	0.949498	0.875773	0.807073	0.748079	0.702811	0.674355
17.5	0.0152778	0.0152693	17	1.526896	1.408339	1.297862	1.202993	1.130198	1.084437
18.5	0.0206111	0.0205997	18	2.059995	1.900045	1.750995	1.623004	1.524794	1.463056
19.5	0.0255	0.0254858	19	2.548594	2.350706	2.166304	2.007956	1.886451	1.810071
20.5	0.0299444	0.0299278	20	2.992793	2.760415	2.543873	2.357926	2.215244	2.125551
21.5	0.0339444	0.0339256	21	3.392592	3.129171	2.883702	2.672915	2.511172	2.409498
22.5	0.0375	0.0374792	22	3.747891	3.456883	3.185707	2.952843	2.774162	2.661839
23.5	0.0406111	0.0405885	23	4.05889	3.743734	3.450056	3.19787	3.004361	2.882718
24.5	0.0432778	0.0432537	24	4.325389	3.989541	3.67658	3.407836	3.201622	3.071992
25.5	0.0455	0.0454747	25	4.547489	4.194395	3.865365	3.582821	3.366019	3.229732
26.5	0.0472778	0.0472515	26	4.725188	4.358297	4.016409	3.722825	3.497551	3.355939
27.5	0.0486111	0.0485841	27	4.858388	4.481154	4.129629	3.827769	3.596144	3.45054
28.5	0.0495	0.0494725	28	4.947188	4.563059	4.205109	3.897731	3.661873	3.513608
29.5	0.0499444	0.0499167	29	4.991688	4.604104	4.242934	3.932791	3.694812	3.545213
30.5	0.0499444	0.0499167	30	4.991688	4.604104	4.242934	3.932791	3.694812	3.545213
31.5	0.0495	0.0494725	31	4.947188	4.563059	4.205109	3.897731	3.661873	3.513608
32.5	0.0486111	0.0485841	32	4.858388	4.481154	4.129629	3.827769	3.596144	3.45054
33.5	0.0472778	0.0472515	33	4.725188	4.358297	4.016409	3.722825	3.497551	3.355939
34.5	0.0455	0.0454747	34	4.547489	4.194395	3.865365	3.582821	3.366019	3.229732
35.5	0.0432778	0.0432537	35	4.325389	3.989541	3.67658	3.407836	3.201622	3.071992
36.5	0.0406111	0.0405885	36	4.05889	3.743734	3.450056	3.19787	3.004361	2.882718
37.5	0.0375	0.0374792	37	3.747891	3.456883	3.185707	2.952843	2.774162	2.661839
38.5	0.0339444	0.0339256	38	3.392592	3.129171	2.883702	2.672915	2.511172	2.409498
39.5	0.0299444	0.0299278	39	2.992793	2.760415	2.543873	2.357926	2.215244	2.125551
40.5	0.0255	0.0254858	40	2.548594	2.350706	2.166304	2.007956	1.886451	1.810071
41.5	0.0206111	0.0205997	41	2.059995	1.900045	1.750995	1.623004	1.524794	1.463056
42.5	0.0152778	0.0152693	42	1.526896	1.408339	1.297862	1.202993	1.130198	1.084437
43.5	0.0095	0.0094947	43	0.949498	0.875773	0.807073	0.748079	0.702811	0.674355
44.5	0.0032778	0.003276	44	0.327599	0.302162	0.278459	0.258105	0.242487	0.232669
	1.0005556	0.9999996	$g(t)$ check	100.0002	92.23556	85.00012	78.78693	74.0194	71.02244

Table F-1: (Continued)

<i>f(x) from</i>	<i>Eq(16)</i>	Time(t)	42	43	44	45	46	47	
		<i>g(t)</i>	70	71.0223	74.01939	78.78701	85.00027	92.23574	
Discrete		C(t-15)	102.1744	101.5375	100.7959	100	99.20413	98.4625	
Midyear	<i>f(x+.5)</i>	Adjusted <i>f</i>							
Age	<i>Eq(15)</i>	Sums to 1	Age	Array of age-specific births					
15.5	0.0032778	0.003276	15	0.22932	0.232669	0.242488	0.258106	0.278461	0.302164
16.5	0.0095	0.0094947	16	0.66465	0.674357	0.702814	0.748083	0.807078	0.875778
17.5	0.0152778	0.0152693	17	1.06883	1.08444	1.130202	1.202999	1.297869	1.408347
18.5	0.0206111	0.0205997	18	1.442	1.463059	1.524799	1.623012	1.751006	1.900056
19.5	0.0255	0.0254858	19	1.78402	1.810074	1.886458	2.007966	2.166317	2.35072
20.5	0.0299444	0.0299278	20	2.09496	2.125555	2.215252	2.357938	2.543888	2.760431
21.5	0.0339444	0.0339256	21	2.37482	2.409503	2.511182	2.672928	2.883719	3.12919
22.5	0.0375	0.0374792	22	2.62353	2.661845	2.774173	2.952858	3.185725	3.456903
23.5	0.0406111	0.0405885	23	2.84123	2.882724	3.004373	3.197886	3.450076	3.743756
24.5	0.0432778	0.0432537	24	3.02778	3.071999	3.201635	3.407853	3.676602	3.989565
25.5	0.0455	0.0454747	25	3.18325	3.229739	3.366032	3.582839	3.865387	4.19442
26.5	0.0472778	0.0472515	26	3.30764	3.355946	3.497564	3.722844	4.016433	4.358323
27.5	0.0486111	0.0485841	27	3.40088	3.450547	3.596158	3.827788	4.129653	4.481181
28.5	0.0495	0.0494725	28	3.46304	3.513615	3.661887	3.897751	4.205133	4.563086
29.5	0.0499444	0.0499167	29	3.49419	3.54522	3.694826	3.932811	4.242958	4.604131
30.5	0.0499444	0.0499167	30	3.49419	3.54522	3.694826	3.932811	4.242958	4.604131
31.5	0.0495	0.0494725	31	3.46304	3.513615	3.661887	3.897751	4.205133	4.563086
32.5	0.0486111	0.0485841	32	3.40088	3.450547	3.596158	3.827788	4.129653	4.481181
33.5	0.0472778	0.0472515	33	3.30764	3.355946	3.497564	3.722844	4.016433	4.358323
34.5	0.0455	0.0454747	34	3.18325	3.229739	3.366032	3.582839	3.865387	4.19442
35.5	0.0432778	0.0432537	35	3.02778	3.071999	3.201635	3.407853	3.676602	3.989565
36.5	0.0406111	0.0405885	36	2.84123	2.882724	3.004373	3.197886	3.450076	3.743756
37.5	0.0375	0.0374792	37	2.62353	2.661845	2.774173	2.952858	3.185725	3.456903
38.5	0.0339444	0.0339256	38	2.37482	2.409503	2.511182	2.672928	2.883719	3.12919
39.5	0.0299444	0.0299278	39	2.09496	2.125555	2.215252	2.357938	2.543888	2.760431
40.5	0.0255	0.0254858	40	1.78402	1.810074	1.886458	2.007966	2.166317	2.35072
41.5	0.0206111	0.0205997	41	1.442	1.463059	1.524799	1.623012	1.751006	1.900056
42.5	0.0152778	0.0152693	42	1.06883	1.08444	1.130202	1.202999	1.297869	1.408347
43.5	0.0095	0.0094947	43	0.66465	0.674357	0.702814	0.748083	0.807078	0.875778
44.5	0.0032778	0.003276	44	0.22932	0.232669	0.242488	0.258106	0.278461	0.302164
	1.0005556	0.9999996	<i>g(t) check</i>	70.00028	71.02258	74.01968	78.78733	85.00061	92.2361

Table F-1: (Continued)

$f(x)$ from	$Eq(16)$	Time(t)	48	49	50	51	52	
Discrete		$g(t)$	100.0003	107.7649	115.0003	121.2134	125.9809	
		$C(t-15)$	97.82565	97.33697	97.02978	96.925	97.02978	
Midyear	$f(x+.5)$	Adjusted f						
Age	$Eq(15)$	Sums to 1	Age	Array of age-specific births				
15.5	0.0032778	0.003276	15	0.327601	0.353038	0.376741	0.397095	0.412714
16.5	0.0095	0.0094947	16	0.949503	1.023228	1.091928	1.150922	1.196189
17.5	0.0152778	0.0152693	17	1.526905	1.645462	1.755939	1.850808	1.923603
18.5	0.0206111	0.0205997	18	2.060007	2.219957	2.369006	2.496997	2.595207
19.5	0.0255	0.0254858	19	2.548608	2.746496	2.930897	3.089246	3.21075
20.5	0.0299444	0.0299278	20	2.99281	3.225188	3.441729	3.627676	3.770358
21.5	0.0339444	0.0339256	21	3.392611	3.656032	3.9015	4.112287	4.274029
22.5	0.0375	0.0374792	22	3.747912	4.03892	4.310096	4.542959	4.72164
23.5	0.0406111	0.0405885	23	4.058913	4.374069	4.667747	4.919933	5.11344
24.5	0.0432778	0.0432537	24	4.325414	4.661263	4.974223	5.242966	5.449179
25.5	0.0455	0.0454747	25	4.547515	4.900608	5.229638	5.512182	5.728983
26.5	0.0472778	0.0472515	26	4.725215	5.092107	5.433994	5.727578	5.952851
27.5	0.0486111	0.0485841	27	4.858416	5.235649	5.587174	5.889034	6.120658
28.5	0.0495	0.0494725	28	4.947216	5.331345	5.689294	5.996672	6.232529
29.5	0.0499444	0.0499167	29	4.991716	5.3793	5.74047	6.050612	6.28859
30.5	0.0499444	0.0499167	30	4.991716	5.3793	5.74047	6.050612	6.28859
31.5	0.0495	0.0494725	31	4.947216	5.331345	5.689294	5.996672	6.232529
32.5	0.0486111	0.0485841	32	4.858416	5.235649	5.587174	5.889034	6.120658
33.5	0.0472778	0.0472515	33	4.725215	5.092107	5.433994	5.727578	5.952851
34.5	0.0455	0.0454747	34	4.547515	4.900608	5.229638	5.512182	5.728983
35.5	0.0432778	0.0432537	35	4.325414	4.661263	4.974223	5.242966	5.449179
36.5	0.0406111	0.0405885	36	4.058913	4.374069	4.667747	4.919933	5.11344
37.5	0.0375	0.0374792	37	3.747912	4.03892	4.310096	4.542959	4.72164
38.5	0.0339444	0.0339256	38	3.392611	3.656032	3.9015	4.112287	4.274029
39.5	0.0299444	0.0299278	39	2.99281	3.225188	3.441729	3.627676	3.770358
40.5	0.0255	0.0254858	40	2.548608	2.746496	2.930897	3.089246	3.21075
41.5	0.0206111	0.0205997	41	2.060007	2.219957	2.369006	2.496997	2.595207
42.5	0.0152778	0.0152693	42	1.526905	1.645462	1.755939	1.850808	1.923603
43.5	0.0095	0.0094947	43	0.949503	1.023228	1.091928	1.150922	1.196189
44.5	0.0032778	0.003276	44	0.327601	0.353038	0.376741	0.397095	0.412714
	1.0005556	0.9999996	$g(t)$ check	100.0007	107.7653	115.0008	121.2139	125.9814

The calculation of the age-time-specific birth array begins by finding the number of period births for times 0 through 52. For each of those years, the births at every age from 15 through 44 are found by multiplying the appropriate $g(t)$ by $f_{x+\frac{1}{2}}$. The calculations are then verified by summing the age-specific births for each year and confirming that they sum to $g(t)$. That verification is shown, beginning in column (5) of Table F-1.

Cohort births are then found by summing diagonal elements of the birth array. Time 0 corresponds to the commencement of fertility for the cohort born in year -15. That

cohort's fertility ends at time 29. Births for 24 consecutive cohorts (one cycle) are followed, through the cohort born at time +8, whose fertility is complete at time 52.

Table F-1 shows total cohort births from Equation (30) on line 3, beginning in column (5), which shows $C(t-15) = 97.82565$. The corresponding total cohort births found by summing array elements is shown on the third line from the bottom of the table, beginning in column (5), where the first entry is 98.13621. The next line shows the percent error (-0.00317) and the bottom line the magnitude or absolute value of the error (0.003175). After time 24, Table F-1 shows that the average absolute error between the Equation (30) value and the array sum value was 0.00256. The magnitude of the largest error was 0.004001. In short, given the inevitable discrepancies in approximating a continuous-time relationship with discrete data, Equation (30) is confirmed by the calculations shown in Table F-1.

Appendix G. Finding the extrema of cohort total fertility in cyclically hyperstationary models

We seek the times within a cycle when the CFR(t) curve of Equation (41),

$$\text{CFR}(t) = \left\{ \frac{a + b \sin \omega(t+30)[\sin 15\omega - 15\omega \cos 15\omega]/(1125\omega^3)}{.5[a + b \sin \omega t]} \right\} \quad (\text{G.1})$$

attains its maximum and minimum values. Following the usual procedure of differentiating the function and setting the derivative to 0, we find

$$\text{CFR}(t)' = 0 = (a+b\sin \omega t)[H(\omega) \cos \omega(t+30)] - [a+H(\omega) \sin \omega(t+30)][b \cos \omega t] \quad (\text{G.2})$$

where, following Equation (31),

$$H(\omega) = [\sin 15\omega - 15\omega \cos 15\omega]/(1125\omega^3)$$

and common factors have been eliminated from Equation (G.2).

To simplify Equation (G.2), expand terms to find

$$aH(\omega) \cos \omega(t+30) + (b H(\omega) \sin \omega t)(\cos \omega(t+30)) = ab \cos \omega t + bH(\omega) \sin \omega(t+30) \cos \omega t \quad (\text{G.3})$$

From Bayer (1978: 168) we have the trigonometric identity

$$\sin x \cos y = \frac{1}{2} \sin(x+y) + \frac{1}{2} \sin(x-y) \quad (\text{G.4})$$

Using Equation (G.4) in Equation (G.2) yields

$$aH(\omega) \cos \omega(t+30) - ab \cos \omega t = bH(\omega) \sin 30\omega \quad (\text{G.5})$$

Equation (G.5) is the result in Equation (42), which can be numerically solved for the time extrema using Maple or a similar mathematical package. The fertility values at those extrema can then be found from Equation (G.1).

