DEMOGRAPHIC RESEARCH
A peer-reviewed, open-access journal of population sciences

DEMOGRAPHIC RESEARCH

VOLUME 50, ARTICLE 11, PAGES 325–346
PUBLISHED 14 FEBRUARY 2024
https://www.demographic-research.org/Volumes/Vol50/11/
DOI: 10.4054/DemRes.2024.50.11

Research Article

Longevity à la mode: A discretized derivative tests method for accurate estimation of the adult modal age at death

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This publication is part of the Special Collection in the Memory of Professor James W Vaupel (1945–2022), founder and long-time publisher of Demographic Research. The Special Collection is edited by Jakub Bijak, Griffith Feeney, Nico Keilman, and Carl Schmertmann.

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Demographic Research: Volume 50, Article 11
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Abstract

BACKGROUND
The modal age at death (or mode) is an important indicator of longevity associated with different mortality regularities. Accurate estimates of the mode are essential, but existing methods are not always able to provide them.

OBJECTIVE
Our objective is to develop a method to estimate the modal age at death, which is purely based on its mathematical properties.

METHODS
The mode maximizes the density of the age-at-death distribution. In addition, at the mode, the rate of aging equals the force of mortality. Using these properties, we develop a novel discrete estimation method for the mode, the discretized derivative tests (DDT) method, and compare its outcomes to those of other existing models.

RESULTS
Both the modal age at death and the rate of aging have been increasing since 1960 in low-mortality countries. The DDT method produces close estimates to the ones generated by the P-spline smoothing.

CONCLUSIONS
The modal age at death plays a central role in estimating longevity advancement, quantifying mortality postponement, and estimating the rate of aging. The novel DDT method proposed here provides a simple and mathematically based estimation of the modal age at death. The method accounts for the mathematical properties of the mode and is not computationally demanding.

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CONTRIBUTION
Our research was motivated by James W. Vaupel, who wanted to find a way to accurately estimate the mode based on its mathematical properties. This article also expands on some of his last research papers that link the modal age at death for populations to the one for individuals.

1. Introduction

> *Over the past half century, and especially in the most recent decades, remarkable improvements have been achieved in survival at older ages, especially at the highest ages. This progress has accelerated the growth of the population of older people and has advanced the frontier of human survival substantially beyond the extremes of longevity attained in preindustrial times.* (Vaupel 1998: 246)

While changes in health and mortality are often studied through changes in life expectancy, under current mortality schedules, changes in longevity have been best described by changes in the modal age at death. Longevity refers to the ability to live longer than the average person. The modal age at death, or mode, captures the most common lifespan, and its changes have been shown to be solely driven by changes in mortality at old ages (Canudas-Romo 2010). More recently, the mode additionally became a measure for disparities at old age (Diaconu, van Raalte, Martikainen 2022), an indicator for analyzing the shifts in mortality (Bergeron-Boucher, Ebeling, and Canudas-Romo 2015), and a forecasting tool (Baselinni and Camarda 2019; Bergeron-Boucher, Vazquez-Castillo, and Missov 2023).

However, the estimation of the modal age at death is, at the very least, challenging given the discrete nature of mortality data. To remedy this problem, previous methods of estimation rely on assumptions about the shape of the lifespan distributions or smoothing techniques. This paper aims to develop a new method to estimate the old-age modal age at death, which is based on its definition and related mathematical properties. The paper is organized as follows. In Section 2, we present a summary of the present research on the modal age at death: its relationship with other mortality measures, current estimation procedures, and observed trends in the mode. In Section 3, data and methods, we introduce the data source and present the mathematical properties of the mode. More importantly, we derive and justify the discretized derivative tests (DDT) method, as well as explain the procedures used for its comparison to other estimates of the mode. In Section 4, we apply the DDT method to six human populations extracted from the Human Mortality Database. Additionally, we estimate the rate of aging by the different methods using the mode estimates for those populations. Finally, in Section 5, we present the discussion of our findings and future lines of research on this topic.

2. Background

The age-at-death distribution is usually a bimodal curve with separate infant and adult mortality peaks. However, the highest peak in low-mortality regimes is the adult (or senescent) one. In this paper, when referring to the modal age at death, we are implying the adult mortality peak. For populations with such a mortality schedule, Kannisto (2001) suggests that there is only one true modal age at death for adult mortality, and if more bumps are observed, it is due to data quality issues. Finding that one and only data point, especially in the presence of other candidates comparable in magnitude and location, is an important and nontrivial task.

The modal age at death has been an increasingly used indicator of longevity in recent years (Ouellette and Bourbeau 2011), but the mode's history in demography traces way back. In his essay on the normal age at death, Lexis (1878) combines two concepts: Quetelet's notion of a "normal man" and the Gaussian (normal) distribution (Véron, Rohrbasser, and Mendelbaum 2003). Lexis posits the existence of a "normal age" in which most normal deaths – non-infant and non-premature – occur, and this age is the center of a Gaussian distribution that describes senescent deaths. Lexis defines the "normal age" as the age at which most deaths occur – the correspondence with the mode is clear – and estimates it between 72 and 73 years for the late 19th-century populations (Robine 2018).

It was not until Kannisto (2001) resumed Lexis's work and started working on the modal age at death that it became a measure of longevity (Robine 2018). In his innovative paper, Kannisto proposes a method for estimating the modal age at death and uses it as a longevity indicator to compare by country and sex. Kannisto also calculates the dispersion of deaths above the modal age at death and compares the mode to other lifespan indicators (Kannisto 2001). Since then, the mode has become one of the most frequently used longevity indicators as (1) it is robust to changes in mortality at early ages, and (2) its almost linear time trend provides a convenient tool to measure the shift of mortality to older ages (Canudas-Romo 2008), as well as forecast adult mortality (Bergeron-Boucher, Vazquez-Castillo, and Missov 2023). Finding the exact location of the mode from empirical data can be problematic given the variation around it and data fluctuations (Horiuchi et al. 2013). Still, given its importance in quantifying and forecasting old-age mortality, it is essential to estimate it accurately.

2.1 The mode's relationship with mortality-related measures

The mode is the maximum of the density function of the distribution of deaths, $d(x)$. It is the age at which the first derivative of $d(x)$ equals zero, and the second derivative of $d(x)$ is negative (the necessary and sufficient condition for having a maximum in a continuous setting). A property of the mode is that the force of mortality at the mode equals its relative derivative. Pollard (1991) derives it assuming a Gompertz model while Canudas-Romo (2008) proves that it holds regardless of the mortality model. The relationship implies that the age-specific rate of mortality change at the mode is the same as the force of mortality at the mode. Ediev (2011) generalizes this finding by deriving a family of formulas that link the force of mortality with its derivative at the mode.

Another relationship links the mode to the survival function (Wilmoth and Horiuchi 1999). The mode is the maximum of the age-at-death distribution, and it equals the maximal downward slope of the survival function, that is the mode corresponds to the point of the fastest decline in the survival curve.

In his last work, Vaupel (2022) describes the importance of the mode for estimating other mortality measures, such as β, the presumably constant (Vaupel 2010) rate of increase with age of the force of mortality for individuals. In a gamma-Gompertz frailty model setting (Vaupel, Manton, and Stallard 1979), he applies the relationship that the force of mortality equals its relative derivative at the mode (Vaupel and Zhang 2010; Vaupel and Missov 2014) to link the rates of aging for populations and individuals.

Previous mode estimation methods apply numerical methods and smoothing techniques rather than their mathematical properties. Here, we propose a simple estimating procedure that takes advantage of the first- and second-derivative tests for a maximum, as well as the property of the force of mortality being equal to its relative derivative at the mode.

2.2 Existing methods for estimating the modal age at death

Given that there is not a straightforward formula for calculating the mode, different methods have been developed for its approximation. In general, one could classify the methods for estimating the modal age at death into two groups: parametric and nonparametric methods. The first type assumes a particular shape of the age-at-death distribution or the shape of the mortality curve and finds through it the modal age at death (Horiuchi et al. 2013). An example of this could be the use of the Gompertz or logistic distributions.

Nonparametric methods, on the other hand, estimate the mode without imposing any shape on the risk of dying. Kannisto's discrete procedure approximates the mode with a quadratic function assumption (Canudas-Romo 2008). Smoothing procedures have also been used to estimate the mode, such as penalized spline (P-spline). Although it is a nonparametric method, P-spline smoothing (Eilers and Marx 1996) yields a continuous force of mortality and aids in estimating the mode more precisely (Ouellette and Bourbeau 2011).

P-spline smoothing approximates the observed death counts by polynomial pieces that are joined in knots (B-splines) and penalizes for the number of selected knots (P-splines). The main advantage of P-spline smoothing over other statistical estimation methods is that it finds the optimal trade-off between parsimony and fitting (Eilers and Marx 1996; Ouellette and Bourbeau 2011). However, P-spline smoothing is computationally demanding, assumes death counts are Poisson distributed, requires choosing the number of knots to be penalized, and yields unsatisfactory fits at the boundaries of the age-at-death distribution (Horiuchi et al. 2013). Moreover, smoothing can also reduce the height of the mode, resulting in changes in its dispersion (Kannisto 2001).

2.3 Trends in the mode

Steady improvements in the age-specific death rates have shifted the mortality schedule and, in particular, the modal age at death toward older ages. Kannisto (2001) observes that the mode is different between males and females, as well as considerably higher than other longevity measures, such as life expectancy. Additionally, he shows that the remaining life expectancy at the mode is inversely proportional to the mode, meaning that when the mode increases, the remaining life expectancy at the mode decreases. He calls this “an invisible wall” that prevents improvements in mortality at old ages.

Using Kannisto’s formula (Kannisto 2001), Canudas-Romo (2008) estimates the upward trend of the mode between 1900 and 2005 for six industrialized countries, while Canudas-Romo (2010) compares the time trajectory of the mode to other longevity measures with a positively increasing trend, such as the maximum age and the median age at death. He also compares their record levels over time. For all studied populations, although with a slightly different slope, Canudas-Romo (2008, 2010) finds a common linearly increasing trend for the mode. Using P-splines and data from four countries, Ouellette and Bourbeau (2011) also find that the estimated mode increases almost linearly over the last decades with different paces for females and males. Thus, the apparently undisturbed linear increase of the mode makes it a remarkably stable demographic indicator that can be used, among others, for forecasting the distribution of adult deaths (Bergeron-Boucher, Vazquez-Castillo, and Missov 2023).

3. Data and methods

3.1 Data

We apply the proposed method to data from the Human Mortality Database (HMD 2023). The HMD is a harmonized data collection that includes information on death and population exposure estimates based on vital registration systems and censuses. For illustrative purposes, in this paper, we use the death counts and exposures for six female populations by single year of age to estimate the age-specific mortality rates and construct the corresponding life tables from 1960 to 2019 (the last year available for all countries). The six populations included in the analysis are the United States, France, Italy, Japan, Denmark, and the Netherlands. This selection of countries allows us to evaluate the method in different mortality regimes and for different population sizes. For Kannisto's method, we use the already-available estimated life tables provided by HMD. For the other methods, we use the observed (non-smoothed) death rates to calculate life tables.

3.2 Mathematical properties of the modal age at death

The mode is the age in which the distribution of deaths reaches its maximum. In addition, as already discussed in Section 2, the force of mortality, $\mu(x)$, at the mode M_o should equal its relative derivative (Canudas-Romo 2008). The latter can also be estimated in the discrete case when it measures the change in the risk of dying between age *x* and *x+1*, also known as the life-table aging rate, or LAR (Horiuchi and Coale 1990; Horiuchi and Wilmoth 1997), and is denoted by *k*(*x*). Using this notation, the finding of Canudas-Romo (2008) can be expressed as

$$k(M_o) = \left.\frac{d \ln \mu(x)}{dx}\right|_{x=M_o} = \mu(M_o) \, . \tag{1}$$

From the properties of derivatives and (1), it follows that

$$\left.\frac{d\mu(x)}{dx}\right|_{x=M_o} = \mu^2(M_o). \tag{2}$$

Although (1), its equivalent (2), and the local maximum conditions (about the first and second derivatives of $d(x)$) provide arithmetic expressions for finding the modal age at death in a continuous setting, the observed death counts and exposures are discrete.

Thus, to estimate the mode from the equations above, we must adjust the death distribution accordingly. Equations (1) and (2) are the cases $\alpha = 0$ and $\alpha = 1$, respectively, in Ediev's (2011) family of equations.

Additionally, the mode is linked to the survival function straightforwardly (Wilmoth and Horiuchi 1999):

$$\begin{aligned} \max d(x) &= \max\{l(x) - l(x+1)\} = \max\{-[l(x+1) - l(x)]\} = \\ &= \max\{-\Delta l(x)\} \approx \max\left\{\frac{-\partial l(x)}{\partial x}\right\}. \end{aligned} \tag{3}$$

3.3 The discretized derivative tests (DDT) method

The mode can be estimated from the observed age-at-death distribution using the mathematical properties described in Section 3.2 with discretized derivatives. We propose approximating the derivative of $d(x)$, the life-table age-at-death distribution, by centered finite differences (for more information about this, read Appendix B), which discretizes the conditions for a local maximum in the following way:

$$d(x+1) - d(x-1) = 0 \tag{4}$$

$$d(x+1) - 2d(x) + d(x-1) < 0. \tag{5}$$

Then, the mode is the age $x = M_o$ that satisfies (4) and (5) simultaneously. In addition, the LAR can also be approximated discretely (Horiuchi and Coale 1990) as

$$k^*(x) = \ln\big(M(x)\big) - \ln\big(M(x-1)\big), \tag{6}$$

where $M(x)$ is the observed age-specific death rate at age x. Furthermore, if we assume a constant force of mortality in the age interval $(x-1, x]$, $M(x)$ equals the life-table age-specific death rate $m(x)$ and the force of mortality $\mu^*(x)$ (Preston, Heuveline, and Guillot 2001: 62). Then, Equation (1) becomes

$$\ln\big(M(M_o)\big) - \ln\big(M(M_o - 1)\big) = M(M_o) \tag{7}$$

while Equation (3) is transformed into

$$\frac{M(M_o + 1) - M(M_o - 1)}{2} = M^2(M_o). \tag{8}$$

Even though (1) and (3) are formally equivalent, having these estimates provides practical convenience in the sensitivity analysis (Ediev 2011); thus, it is good to consider both to account for minor estimation errors. Conversely, the expression from (3) is an exact derivation from finding the maximum at $d(x)$, thus it is not necessary to include it as a condition (as it completely overlaps $d(x)$). As a result, the mode must satisfy simultaneously the conditions in (4), (5), (7), and (8), which can be rewritten as

$$\left.\begin{array}{lr} d(M_o + 1) - \mathrm{d}(\mathrm{M_o} - 1) = 0 & (9.1) \\ \mathrm{d}(M_o + 1) - 2d(\mathrm{M_o}) + \mathrm{d}(\mathrm{M_o} - 1) < 0 & (9.2) \\ \mathrm{k}^*(\mathrm{M_o}) - \mathrm{M}(\mathrm{M_o}) = 0 & (9.3) \\ \mathrm{M}(\mathrm{M_o} + 1) - \mathrm{M}(\mathrm{M_o} - 1) - 2\mathrm{M}^2(\mathrm{M_o}) = 0. & (9.4) \end{array}\right\} \tag{9}$$

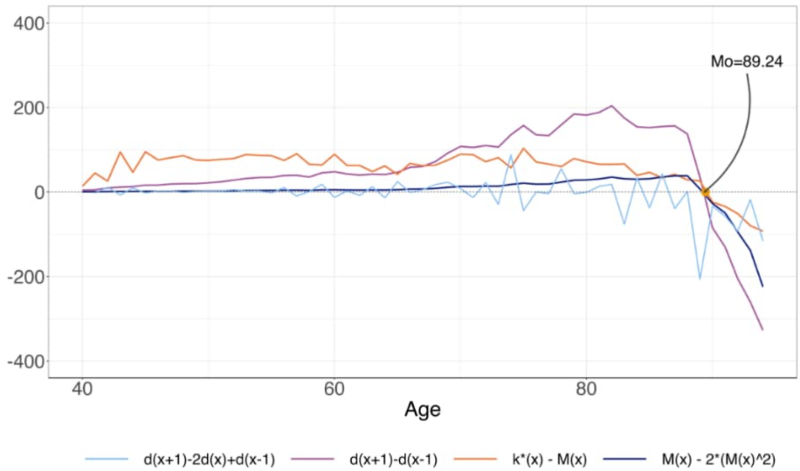
Given that the ages in the data we are using are integers, we might not be able to identify the exact M_o values that satisfy all conditions above. To handle this, we identify the two consecutive (integer) M_o values for which the conditions in (9) change from positive to negative – that is, we determine the age interval in which each left-hand side in (9) turns zero. Then, for each of the three conditions, considering being a maximum as one condition, we interpolate between the endpoints of this interval, then calculate the average of these estimates and use it as an estimate of the mode. We will refer to this estimation procedure by the DDT method.

The conditions in (9) can be met at one point or at multiple points given that they hold for any local maximum. Therefore, after finding all possible solutions to (9), we select the mode as the value for which $d(x)$ is the highest. It is important to remember that condition 9.2 is met by just being negative. This means that it serves as a validation point and does not provide an estimate of the mode on its own.

Figure 1 illustrates the case when all four conditions in (9) are met just once and visibly at one and the same point. Here, we estimate the mode as the average of the three estimated (very close to one another) zero-crossing points for each of the lines (each line corresponds to an expression on the left-hand side of (9)).

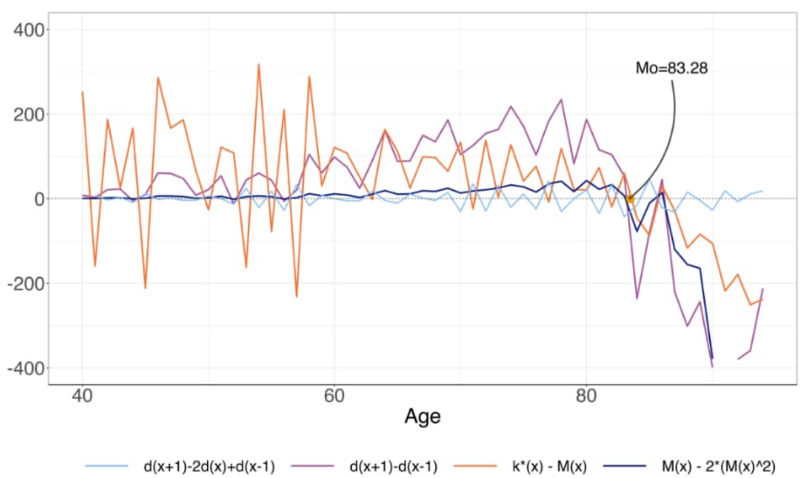
In Figure 2, though, we can observe multiple intersection points along the age axis for the left-hand sides of the conditions with zero. All these points correspond to local maxima that satisfy (9). Here, the mode is determined as the age that fulfills the four properties and has the highest number of life-table death counts.

Figure 1: Conditions in Equation (9) by age and the modal age-at-death estimation based on its mathematical properties for females in 2019 in the United States



Sources: HMD (2023) and authors' own calculation.

Figure 2: Conditions in Equation (9) by age and the modal age-at-death estimation based on its mathematical properties for females in 1972 in Denmark



Sources: HMD (2023) and authors' own calculation.

3.4 Comparison with other models

We compared the performance of the DDT method with the three previously applied estimation methods for the mode: the Kannisto (nonparametric, discrete), Gompertz (parametric, continuous) methods, and P-spline smoothing (nonparametric, continuous).

The Kannisto (2001) formula defines the mode as

$$M_o = x + \frac{d(x) - d(x-1)}{\big(d(x) - d(x-1)\big) + \big(d(x) - d(x+1)\big)}. \tag{10}$$

In a parametric model setting, several closed-form expressions for the modal age at death have been derived. For example, Canudas-Romo (2008) presents an exact formula for the mode in a Gompertz and a logistic setting, while Missov et al. (2015) express it also for the gamma-Gompertz and Weibull distributions (Table 1). However, not every model has an analytical expression for the modal age at death. We select the Gompertz model as an example of a parametric mortality model to apply because it provides closed-form expressions for most of the quantities in (9) and fulfills all mathematical properties of the mode. The proof for the latter is straightforward. On the one hand,

$$\frac{d}{dx}\ln(\mu(x)) = \frac{d}{dx}\ln(ae^{bx}) == \frac{d}{dx}(\ln(a) + bx) = b. \tag{11}$$

On the other hand, using the formula for the Gompertz modal age at death (Canudas-Romo 2008; Missov et al. 2015),

$$\mu(M_o) = ae^{b\cdot\frac{\ln\left(\frac{b}{a}\right)}{b}} = a\cdot\frac{b}{a} = b. \tag{12}$$

Therefore, when $\mu(x)$ is a Gompertz force of mortality, (1) holds – that is,

$$\frac{d}{dx}\ln\big(\mu(x)\big) = b = \mu(M_o). \tag{13}$$

Table 1: Parametrical modal age at death for the models Gompertz, logistic, gamma-Gompertz, and Weibull

Distribution	$\mu(x)$	Modal age at death (M_o)
Gompertz [1,2]	ae^{bx}	$M_o = \frac{1}{b} \cdot \ln\frac{b}{a}$
Logistic [1]	$\frac{e^{ax+bxa}}{1 + e^{ax+bxa}}$	$M_o = \frac{\ln(b) - a}{b}$
Gamma-Gompertz [2]	$\frac{kae^{bx}}{\lambda + \frac{a}{b}(e^{bx} - 1)}$	$M_o = \frac{1}{b} \cdot \ln\frac{\lambda b - a}{ka}$
Weibull [2]	$\frac{a}{b^a}x^{a-1}$	$M_o = b\left(1 - \frac{1}{a}\right)^{\frac{1}{a}}$

Notes: [1] Canudas-Romo 2008; [2] Missov et al. 2015.

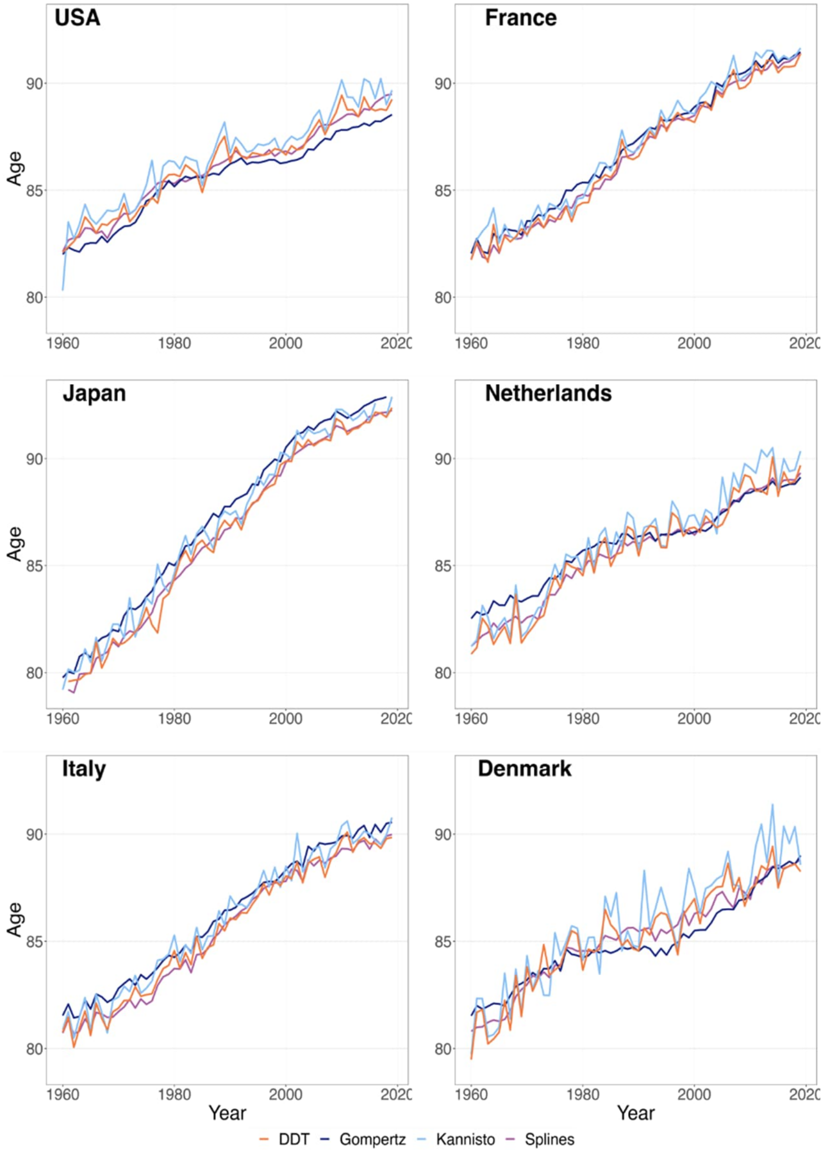
We apply P-spline smoothing using the R package *MortalitySmooth* (Camarda 2012). We first smooth the death counts and then calculate the respective age-specific death rates, life-table death counts, and finally, M_o.

4. Results

Figure 3 shows the mode estimates for females in the six studied countries by each of the four estimation methods described earlier. The estimates from the DDT model closely follow the ones from P-spline smoothing. However, the mode patterns from the DDT model are more erratic, as are the Kannisto's, given that they are based on (non-smoothed) discrete data. Figure 3 shows that the Kannisto model tends to overestimate the mode.

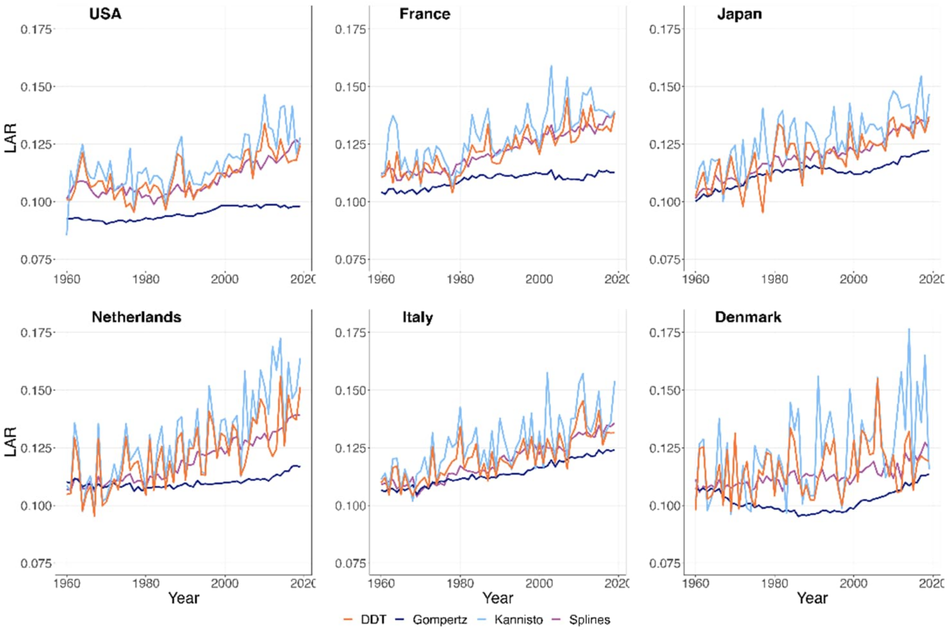
As shown in (1) and (13), the modal age at death can be used to find the population rate of aging (LAR) as the relative derivative of the force of mortality. Thus, by using the M_o estimates, we can calculate LAR as $k(M_o) = \mu(M_o)$. Figure 4 compares the $\mu(M_o)$ estimates across countries by each of the four models. All of them show an increase in LAR over time. A hypothesis, expressed by Vaupel (2010), postulates that individuals might share a common constant rate of aging. While LAR estimates provide only an approximation for the individual rate of aging, the latter reaches values around 0.14 in recent years in most countries, based on the DDT, P-spline, and Kannisto models. The Gompertz model tends to provide lower estimates compared with the other models, which might be attributed to a poor fit of the model to the data.

Figure 3: Modal age-at-death estimates by four methods for six countries, females, 1960–2019



Sources: HMD (2023) and authors' own calculation.

Figure 4: LAR estimates via M_o (four methods), females, 1960–2019



Sources: HMD (2023) and authors' own calculation.

5. Discussion and future lines of research

We suggest a new method to estimate the mode solely based on its mathematical properties. The method provides consistent estimates with other models while not making assumptions on the shape of mortality and is less computationally demanding in comparison to the other models.

When using a parametric model to estimate the modal age at death, the estimated value is highly affected by the goodness of fit. The nonparametric methods we apply have certain shortcomings too. While the Kannisto approach is not so challenging in terms of calculation, it relies mainly on an identified single age, which is not always easily identifiable in observed data given its shaky nature. On the other hand, P-spline smoothing assumes a Poisson distribution for the death counts at each age, which leads to the assumption that the expected value equals the variance. This is rarely the case when we calculate the mean and variance of the (empirical) distribution of deaths. Additionally, P-splines do not control for the validity of the mathematical relationships holding at the mode. In this paper, we offer an alternative for estimating the modal age at death with decimal precision while avoiding some of the limitations of the other models.

One of the limitations of the suggested discretized derivative tests method is that it provides more fluctuating time trends than other methods, such as the P-splines. However, this is expected due to the discrete nature of the data. As a result of this variability, the method is also unstable when analyzing small populations. Since the method yields results similar to P-spline smoothing, it can be argued that the DDT model provides the simplest and most accurate solution in most cases. However, if the studied population is too small, the P-spline approach is preferable. We recommend applying the DDT method only when the data are disaggregated into one-year age intervals. If the gap between the observations is small, it will produce accurate estimates that are, in addition, closer to the ones in the continuous case.

The modal age at death has been considered one of the most important longevity indicators and recently even a disparity indicator (Diaconu, van Raalte, Martikainen 2022). The mode is also instrumental in the study of population aging and the theory of heterogeneity. Vaupel (2022) shows that the modal age at death can be instrumental in estimating other mortality characteristics, such as the level of the late-life mortality plateau. Precise M_o estimates based on the DDT method can also contribute to the testing of the constant rate of aging hypothesis (Vaupel 2010) as well as to checking other mode-related regularities, such as the constant ratio (= 1.23) between the standard deviation above the mode and life expectancy at the mode (Thatcher et al. 2010). Whether these are in fact regularities or artifacts of the model used to estimate the modal age at death should be tested.

The relationships presented here are not novel, but they unveil a simple procedure to estimate the modal age at death with high precision, a natural fit to James W. Vaupel's "model simple, think complex" paradigm. We believe that more accurate M_o estimates will lead to a better assessment of mortality dynamics, longevity extension, and all related survival theories.

6. Acknowledgments

This paper is dedicated to and inspired by James W. Vaupel. We would like to thank the editor and the two anonymous reviewers for their insightful comments that substantially improved our first version. The research and publication of this study were supported by the AXA Research Fund through the funding for the AXA Chair in Longevity Research. MPBB and TIM received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement 884328 – Unequal Lifespans).

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Appendix A: On the relationship of the mode and life-table aging rate

The modal age at death is located at the age in which the relative derivative of the force of mortality equals the force of mortality. The relative derivative of the force of mortality is also known as the life-table aging rate, LAR (Horiuchi and Coale 1990; Horiuchi and Wilmoth 1997). Pollard (1991) derives the relationship between the mode and LAR in a Gompertz setting, while Canudas-Romo (2008) shows that it holds for any arbitrary survival model.

Since the modal age at death is the age at which density of the age-at-death distribution, $d(x)$, is the highest, we set the first derivative of $d(x)$ equal to zero and check if its solution leads to a negative value of the second derivative of $d(x)$. Formally, we have

$$d(x) = \mu(x) * e^{-\int_0^x \mu(a)da}, \tag{A1}$$

and then

$$\begin{aligned} \frac{\partial d(x)}{\partial x} &= \frac{\partial \mu(x)}{\partial x} * e^{-\int_0^x \mu(a)da} - \mu(x) * \mu(x) * e^{-\int_0^x \mu(a)da} \\ &= \left(\frac{\partial \mu(x)}{\partial x} - \mu(x) * \mu(x)\right) * e^{-\int_0^x \mu(a)da} \\ &= \left(\frac{\frac{\partial \mu(x)}{\partial x}}{\mu(x)} - \mu(x)\right) * e^{-\int_0^x \mu(a)da} * \mu(x) \\ &= \left(\frac{\partial \ln \mu(x)}{\partial x} - \mu(x)\right) * d(x). \end{aligned} \tag{A2}$$

Setting (A2) as being equal to zero leads to a solution we will denote by $x = M_0$ such that

$$\frac{\partial \ln \mu(x)}{\partial x} = \mu(x)\Big|_{x=M_0}, \tag{A3}$$

as $d(Mo) > 0$. Checking that the second derivative for $x = M_0$ is also straightforward.

Appendix B: Notes on the finite estimation of derivatives

From finite calculus, there are three different ways to approximate a derivative. These three approximations can be derived from the Taylor series approximation of the function. The formulas for these approaches are as follows:

Forward difference: $$\frac{\partial f(x)}{\partial x} \approx \Delta f(x) = \frac{f(x+h) - f(x)}{h}, \tag{B1}$$

Backward difference: $$\frac{\partial f(x)}{\partial x} \approx \Delta f(x) = \frac{f(x) - f(x-h)}{h}, \tag{B2}$$

Central (centered) difference: $$\frac{\partial f(x)}{\partial x} \approx \Delta f = \frac{f - f}{2h}. \tag{B3}$$

The main difference between B1, B2, and B3 is the error term. For B1 and B2, the error term will be of the order of h, whereas for B3 the error term is of the order of h^2. As h is desirable to be as small as possible, it is clear that an error term of h^2 is preferred, and thus centered differences produce better estimates. In the case of our mortality estimates, h = 1, thus the differences between the three approaches are minimal. Still, we used the centered-differences approach to derivatives.

Thus, in the case of the estimation of the maximum of the age at death distribution,

$$\left.\frac{dd(x)}{dx}\right|_{x=M_o} = 0, \tag{B4}$$

is estimated as:

$$\frac{d(M_o + 1) - \mathrm{d}(\mathrm{M_o} - 1)}{2 \cdot 1} = 0 \tag{B5}$$

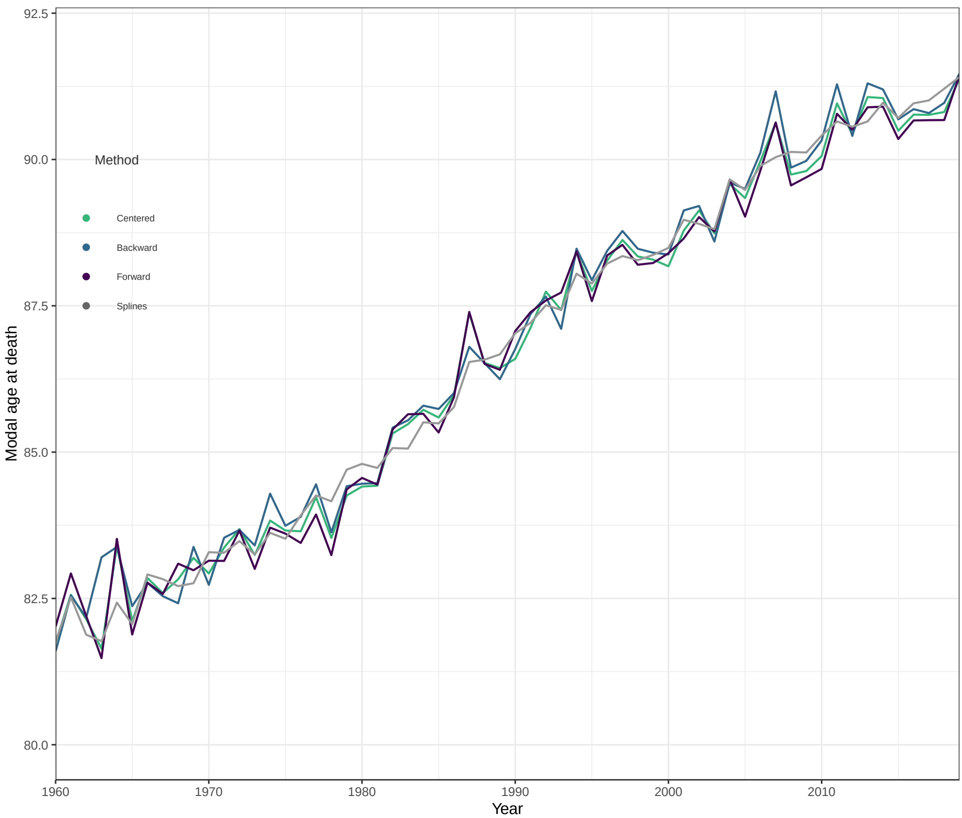
then,

$$d(M_o + 1) - \mathrm{d}(\mathrm{M_o} - 1) = 0, \tag{B6}$$

as shown in equation 9.1.

In Figure A-1, we show the differences in the estimates of the mode for the three different finite derivatives approaches for France and the P-splines estimates for reference. We can see that although the differences are small among the approximations, the centered one produces results closer to the P-splines smoothing. It is important to highlight that the properties in Equation (9) were estimated using the centered approach, except for the second property (9.2) because the discrete LAR has been previously defined by its authors (Horiuchi and Coale 1990).

Figure A-1: M estimates with three methods of approximation (and P-splines) for French females, 1960–2019

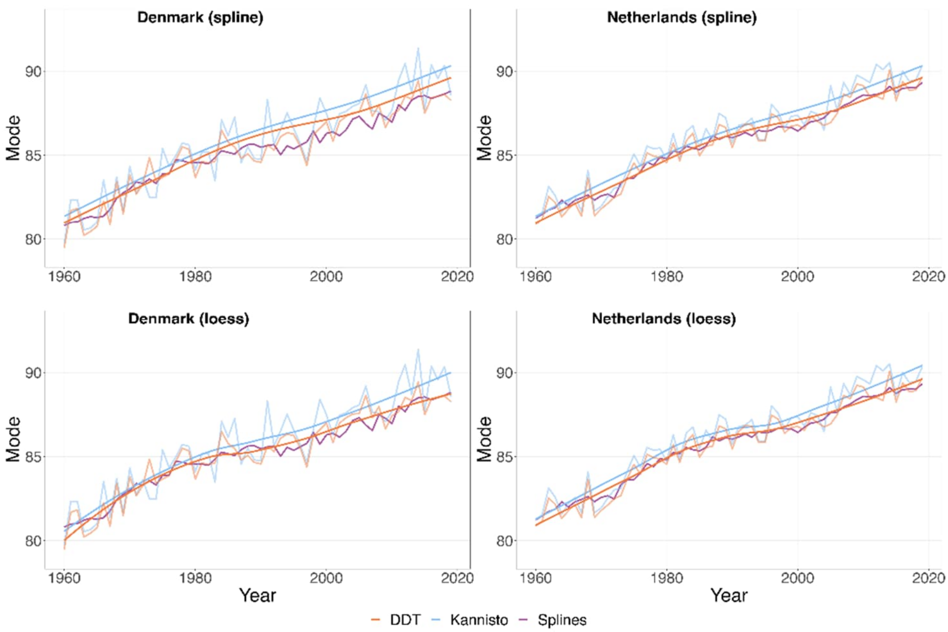


Sources: HMD (2023) and authors' own calculation.

Appendix C: Comparison of the methods under smoothing

One of the mentioned advantages of the DDT method is that it is straightforwardly derived from mortality functions. However, that same advantage makes the method more erratic compared to other estimation procedures, such as P-splines smoothing. Thus, in this appendix, we present the estimates of the method when smoothed and compared with the results of the smoothed Kannisto and the P-splines for Denmark and the Netherlands (the 'bumpiest' in the original estimates). We used spline smoothing from the package *pspline* (Ramsey and Ripley 2022) and *ggplot 2* loess smoothing (Wickham 2016) for this comparison. For both cases, we observe (Figure A-2) that the DDT method estimates are the closest to the P-splines approach and that this is consistent across smoothing methods.

Figure A-2: Smoothed estimates for Denmark and the Netherlands



Sources: HMD (2023) and authors' own calculation.