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*Formal Relationships 1*

# Life lived equals life left in stationary populations

**Joshua R. Goldstein**

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Guest Editors are Joshua R. Goldstein and James W. Vaupel.

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# Life lived equals life left in stationary populations

**Joshua R. Goldstein** [1, 2]

## Abstract

The average age of the individuals in a population is equal to the average remaining life expectancy when the population is stationary.

## 1. Relationship

The mean age of a stationary population is equal to the average remaining life expectancy. Let $a$ denote age, $e(a)$ be life expectancy at age $a$, and $\ell(a)$ be the proportion surviving to age $a$. In a stationary population, the density of individuals aged $a$ is $\ell(a)/\, e(0)$.

If we define the mean life lived and the mean life left as, respectively,

(1) $$\bar{a} = \frac{\int_0^\omega a\,\ell(a)\,da}{e(0)}$$

and

(2) $$\bar{e} = \frac{\int_0^\omega e(a)\,\ell(a)\,da}{e(0)}\,,$$

then

(3) $$\bar{a} = \bar{e}.$$

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[2] I thank James W. Vaupel and an anonymous reviewer.

## 2. Proof

Since life expectancy at age $a$ is $e(a) = \int_a^\omega \ell(x)\,dx/\,\ell(a)$,

$$\bar{e} = \frac{\int_0^\omega \ell(a)\left(\int_a^\omega \ell(x)/\ell(a)\,dx\right)da}{e(0)} = \frac{\int_0^\omega \int_a^\omega \ell(x)\,dx\,da}{e(0)}. \tag{4}$$

Reversing the order of integration, with the corresponding change in the limits of integration, gives

$$\bar{e} = \frac{\int_0^\omega \left(\int_a^\omega \ell(x)\,dx\right)da}{e(0)} = \frac{\int_0^\omega \ell(x)\left(\int_0^x 1\,da\right)dx}{e(0)} = \frac{\int_0^\omega x\,\ell(x)\,dx}{e(0)} = \bar{a}. \tag{5}$$

Q.E.D.

## 3. History and related results

Kim and Aron (1989) gave the same result using integration by parts. A more general form of the result is known from the distributional equality of backward and forward recurrences in renewal theory: see Cox (1962). Keyfitz (1985:74) calls $\bar{e}$ the average expectation of life, denoting it $E$.

It is also possible to show that in a growing stable population $\bar{e} > \bar{a}$, and in a shrinking stable population $\bar{e} < \bar{a}$.

The mean age of a stationary population can be expressed in terms of the mean age of death and the coefficient of variation (CV) of age of death,

$$\bar{a} = \frac{e(0) + \frac{\sigma^2}{e(0)}}{2} = \frac{e(0)}{2}\,(1 + CV^2), \tag{6}$$

where $\sigma^2$ is the variance of age of death and $CV$ is $\sigma/\,e(0)$.

Thus, average remaining life expectancy $\bar{e}$ can also be expressed this way, showing the dependence on the mean and variability of life's length.

## 4. Applications

Time lived and time left are central to debates about population aging. Increases in $\bar{a}$ are seen as making the population"older"; but increasing distance from death $\bar{e}$ is seen as making the population effectively younger (Ryder 1975; Miller 2001). The equality of $\bar{a}$ and $\bar{e}$ in stationary populations means that mortality decline[3] increases both by the

[3]More precisely, mortality decline that does not influence the birth rate.

same amount. As illustrated by Sanderson and Scherbov (2005), populations can even be getting older and younger at the same time, depending on one's point of view. The quantity $\bar{a}/(\bar{a} + \bar{e})$ can be used as a measure of relative aging, varying between 0 and 1, taking the value $\frac{1}{2}$ in stationary populations.

Applications of equality (2) include estimation of $\bar{a}$ when $\bar{e}$ is more easily observed and vice versa. Kim and Aron give an example in epidemiological modeling. A potential application for household demography would be to infer average time remaining in the household from retrospective estimates of average time in the household.

Note: On Tuesday, March 3, 2009, three small typos were corrected in this PDF, on pages 3 and 4.

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