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Formal Relationships 12

Life expectancy: Lower and upper bounds from surviving fractions and remaining life expectancy

Joel E. Cohen

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Life expectancy: Lower and upper bounds from surviving fractions and remaining life expectancy

Joel E. Cohen [1]

Abstract

We give simple upper and lower bounds on life expectancy. In a life-table population, if $e(0)$ is the life expectancy at birth, M is the median length of life, and $e(M)$ is the expected remaining life at age M, then $(M + e(M))/2 \leq e(0) \leq M + e(M)/2$. In general, for any age x, if $e(x)$ is the expected remaining life at age x, and $\ell(x)$ is the fraction of a cohort surviving to age x at least, then $(x + e(x)) \cdot \ell(x) \leq e(0) \leq x + \ell(x) \cdot e(x)$. For any two ages $0 \leq w \leq x \leq \omega$, $(x - w + e(x)) \cdot \ell(x)/\ell(w) \leq e(w) \leq x - w + e(x) \cdot \ell(x)/\ell(w)$. These inequalities give bounds on $e(0)$ even without detailed knowledge of the course of mortality prior to age x, provided $\ell(x)$ can be estimated. Such bounds could be useful for estimating life expectancy when the input of eggs or neonates can be estimated but mortality cannot be observed before late juvenile or early adult ages.

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1. Relationship

The life table $\ell(x)$, constant in time, with continuous age x, is the proportion of a cohort (whether a birth cohort or a synthetic period cohort) that survives to age x or longer. The maximum possible age ω may be finite or infinite. By definition, $\ell(0) = 1$ and $\ell(\omega) = 0$ and $\ell(x)$ is non-increasing with increasing x, and therefore is nonnegative since $\ell(\omega) = 0$. Assume $\ell(x)$ is a continuous function of x, $0 \leq x \leq \omega$. The complete expectation of life at age x, $e(x)$, is the average number of years remaining to be lived by those who have attained age x.

We show here that $e(0)$, life expectancy at birth, satisfies, for every age x,

$$(x + e(x)) \cdot \ell(x) \leq e(0) \leq x + \ell(x) \cdot e(x). \tag{1}$$

More generally, for any two ages $0 \leq w \leq x \leq \omega$,

$$(x - w + e(x)) \cdot \frac{\ell(x)}{\ell(w)} \leq e(w) \leq x - w + e(x) \cdot \frac{\ell(x)}{\ell(w)}. \tag{2}$$

When $w = 0$, (2) reduces to (1). All these inequalities reduce to equality when $x = 0$ in (1) or $x = w$ in (2) and are strict inequalities if $x > 0$ or $x > w$ and $\ell(x)$ is strictly decreasing with increasing x. Subtracting the lower bound in (1) from the upper bound in (1) gives the difference $(x - w)(1 - \ell(x)/\ell(w))$. Both factors in this expression increase with the difference in ages $x - w$, so the closer the age difference $x - w$ is to 0, the closer the lower and upper bounds are to $e(w)$. Also, the higher the probability $\ell(x)/\ell(w)$ of survival from age w to x, the closer the lower and upper bounds are to $e(w)$.

The inequalities take a particularly simple form when $w = 0$ and $x = M$, the median length of life. Then $\ell(M) = 1/2$ and

$$\frac{M + e(M)}{2} \leq e(0) \leq M + \frac{e(M)}{2}. \tag{3}$$

Similarly, when $w = 0$ and $x = U$, the upper quartile of length of life, then $\ell(U) = 1/4$ and

$$\frac{U + e(U)}{4} \leq e(0) \leq U + \frac{e(U)}{4}. \tag{4}$$

Also, from (2),

$$\frac{U - M + e(U)}{2} \leq e(M) \leq U - M + \frac{e(U)}{2}. \tag{5}$$

As a consequence of (1),

$$\sup((x + e(x)) \cdot \ell(x) | 0 \leq x \leq \omega) \leq e(0) \leq \inf(x + \ell(x) \cdot e(x) | 0 \leq x \leq \omega). \tag{6}$$

These inequalities make it possible to estimate bounds for $e(0)$ without detailed knowledge of the course of mortality prior to age x, provided that $\ell(x)$ can be estimated. Such bounds could be useful in estimating life expectancy when the input of eggs or neonates can be estimated but mortality cannot be observed before late juvenile or early adult ages. The lower the mortality before late juvenile or early adult ages, the more closely the bounds will bracket $e(0)$.

2. Proof of (2)

A standard formula (Keyfitz 1968:6) for life expectancy at any age w, $0 \leq w \leq \omega$, is

(7) $$e(w) = \frac{1}{\ell(w)} \int_{a=w}^{a=\omega} \ell(a)da.$$

Therefore, for any ages $w < x$,

$$e(w) = \frac{1}{\ell(w)} \int_{a=w}^{a=x} \ell(a)da + \frac{1}{\ell(w)} \int_{a=x}^{a=\omega} \ell(a)da = T1 + T2.$$

Since $\ell(x)$ is non-increasing as x increases,

(8) $$T1 \geq \frac{1}{\ell(w)} \int_{a=w}^{a=x} \ell(x)da = \frac{\ell(x)}{\ell(w)} \cdot (x - w).$$

(9) $$T2 = \frac{1}{\ell(w)} \int_{a=x}^{a=\omega} \ell(a)da = \frac{1}{\ell(w)} \frac{\ell(x)}{\ell(x)} \int_{a=x}^{a=\omega} \ell(a)da = \frac{\ell(x)}{\ell(w)} e(x).$$

Hence $e(w) \geq (x - w + e(x)) \cdot \ell(x)/\ell(w)$, which proves the lower bound in (2). To prove the upper bound in (2), observe that

(10) $$T1 = \frac{1}{\ell(w)} \int_{a=w}^{a=x} \ell(a)da \leq \frac{1}{\ell(w)} \int_{a=w}^{a=x} \ell(w)da = x - w.$$

Hence $e(w) \leq x - w + e(x) \cdot \ell(x)/\ell(w)$.
Q.E.D.

3. Application to life table of the United States in 2004

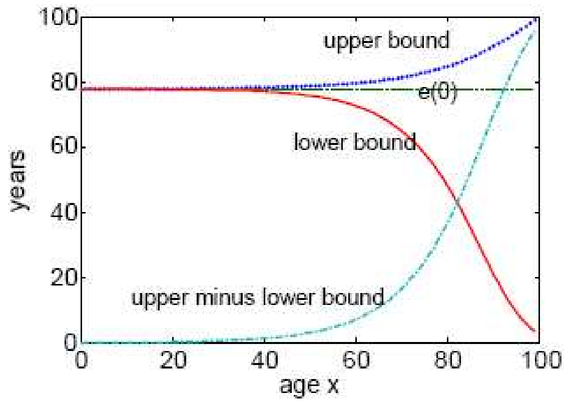
Arias (2007) tabulated ℓ_x, q_x, and e_x for exact ages $0, 1, 2, \ldots, 99$, and a terminal catch-all group 100 years or older, for the 2004 United States population. Table 1 compares the life expectancy at birth $e(0)$ with the upper and lower bounds obtained from (1) at selected ages, and Figure 1 shows the comparison at every age. The difference between the lower and upper bounds is less than 1.4 years up to age 40 and then increases rapidly with increasing age. At every age up to 40 years, the upper bound minus $e(0)$ is greater than or equal to $e(0)$ minus the lower bound (excepting ages 35 and 38, where the exceptions are probably artifacts of rounding), while at every age from 41 years onward, the upper bound minus $e(0)$ is strictly less than $e(0)$ minus the lower bound. That is, the lower bound approximates $e(0)$ more closely at younger ages (up to 40 years), while the upper bound approximates $e(0)$ more closely at older ages in this example.

Table 1: Lower and upper bounds on life expectancy at birth and difference between the bounds, at ages 20, 40, 60, and 80 years, compared to the life expectancy at birth $\mathbf{e(0) = 77.8}$ years (value of the upper and lower bounds for age 0), in the United States' 2004 total population

age	lower bound	upper bound	upper minus lower bound
0	77.8	77.8	0.0
20	77.8	78.0	0.3
40	77.1	78.5	1.4
60	72.6	79.8	7.2
80	48.0	84.9	36.9

Notes: Values of $\mathbf{e(x)}$ and $\boldsymbol{\ell}\mathbf{(x)}$ are taken from Arias (2007); bounds and difference are calculated here from (1).

Figure 1: For the United States total population in 2004, for every age x from 0 to 99 years, upper bounds (rising dark blue dots), lower bounds (falling red solid line), the difference between the upper and the lower bounds (rising cyan dash-dot line), and the life expectancy at birth $\mathrm{e}(0)$ (horizontal olive dotted straight line), based on applying (1) to $\ell(\mathrm{x})$ and $\mathrm{e}(\mathrm{x})$ from Arias (2007)



4. Application to the exponential distribution

If the life table is negative exponential with parameter K, i.e., $\ell(x) = \exp(-Kx)$, then $e(x) = 1/K$ for every x. The inequalities (1) become

$$\left(x + \frac{1}{K}\right) e^{-Kx} \leq \frac{1}{K} \leq x + \frac{e^{-Kx}}{K}.$$

These inequalities are easily proved without reference to the general case, as follows. At $x = 0$, both inequalities are equalities. It is elementary to check that for $x > 0$ the derivative of the lower bound (as a function of x) is $-Kxe^{-Kx}$, which is negative, and the derivative of the upper bound is $(1 - e^{-Kx})$, which is positive, so that strict inequalities hold for any $x > 0$.

A referee conjectured that, for the exponential distribution, if $x > 0$, the error of the lower bound, namely, $\frac{1}{K} - (x + \frac{1}{K})e^{-Kx}$, is strictly less than the error of the upper bound, namely, $x + \frac{e^{-Kx}}{K} - \frac{1}{K}$. By simple algebra, the referee observed, the conjectured inequality holds if and only if $f(x) > 0$ for all $x > 0$, where

$$f(x) = x - (2/K)(1 - \exp(-Kx))/(1 + \exp(-Kx)).$$

To prove that $f(x) > 0$ when $x > 0$, we observe that $f(0) = 0$ and $f(x) = x - \left(\frac{2}{K}\right)\tanh\left(\frac{Kx}{2}\right)$. Because $\frac{df(x)}{dx} = (\tanh\left(\frac{Kx}{2}\right))^2 \geq 0$ and this last inequality is strict if $x > 0$, it follows that $f(x) > 0$ when $x > 0$. This proves that the error of the lower bound is less than the error of the upper bound for the exponential distribution when $x > 0$. Incidentally, since $\tanh(x) + \tanh(-x) = 0$, these same calculations show that $f(x) < 0$ when $x < 0$.

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References

Arias, E. (2007). United States Life Tables, 2004. *National Vital Statistics Reports* 56(9). December 28. Hyattsville, MD: National Center for Health Statistics.

Keyfitz, N. (1968). *Introduction to the Mathematics of Population*. Reading, MA: Addison-Wesley.