Research Article

**A dynamic birth-death model
via Intrinsic Linkage**

Robert Schoen

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Robert Schoen[1]

Abstract

BACKGROUND

Dynamic population models, or models with changing vital rates, are only beginning to receive serious attention from mathematical demographers. Despite considerable progress, there is still no general analytical solution for the size or composition of a population generated by an arbitrary sequence of vital rates.

OBJECTIVE

The paper introduces a new approach, Intrinsic Linkage, that in many cases can analytically determine the birth trajectory of a dynamic birth-death population.

METHODS

Intrinsic Linkage assumes a weighted linear relationship between (i) the time trajectory of proportional increases in births in a population and (ii) the trajectory of the intrinsic rates of growth of the projection matrices that move the population forward in time. Flexibility is provided through choice of the weighting parameter, *w*, that links these two trajectories.

RESULTS

New relationships are found linking implied intrinsic and observed population patterns of growth. Past experience is "forgotten" through a process of simple exponential decay. When the intrinsic growth rate trajectory follows a polynomial, exponential, or cyclical pattern, the population birth trajectory can be expressed analytically in closed form. Numerical illustrations provide population values and relationships in metastable and cyclically stable models. Plausible projection matrices are typically found for a broad range of values of *w*, although *w* appears to vary greatly over time in actual populations.

CONCLUSIONS

The Intrinsic Linkage approach extends current techniques for dynamic modeling, revealing new relationships between population structures and the changing vital rates that generate them.

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[1] E-mail: rschoen309@att.net.

1. Introduction

Over the last century, the stable population has been the dominant mathematical model in demography. It is based on unchanging age-specific rates of birth and death and the constant growth and age structures those rates produce. Of course, in actual populations, age-specific vital rates change over time, often substantially. The ability to model such changes in behavior, and to analytically determine their implications for the size and age structure of a population over time, remains a challenge for mathematical demography.

Previous work has explored dynamic models, that is models with time-varying vital rates. Skellum (1967) was the first to analyze cyclical stability. Coale's (1972) pioneering book examined the implications of changing rates in birth-death models. Lee (1974) analyzed interactions between population stocks and flows with a focus on cyclical dynamics. Recent decades have seen increasing interest in dynamic models. Tuljapurkar (1990) examined population change in the context of variable environments. Bongaarts and Feeney (2002) advanced a flexible dynamic mortality-only model. Schoen and Kim (1994) and Schoen and Jonsson (2003) developed the metastable (or quadratic hyperstable) model, generalizing the stable model by allowing fertility to change exponentially over both age and time. Schoen (2006, Chapter 7) discussed several other approaches to modeling changing rates, including Intrinsically Dynamic and "hyperstable" models. Yet despite the progress to date, there is no generally applicable analytical solution for the age structure produced by an arbitrarily changing set of vital rates.

The present effort is not a comprehensive solution but rather a step in this direction. It explores dynamic populations using discrete, time-varying population projection (Leslie) matrices. A new approach, called Intrinsic Linkage, is advanced in that it is applicable to a wide range of vital rate trajectories. The approach leads to analytical population projections and to new, analytically tractable relationships between the time trajectory of Leslie matrix intrinsic growth rates and the birth trajectory associated with it.

2. The basic birth-death (Leslie) model

Consider an n-age group discrete birth-death population model, and let A_t be the population projection (Leslie) matrix that takes the model population from time $t-1$ to time t. Models with 3-age groups are used to facilitate the presentation without loss of generality. To simplify the dynamic relationships, populations are assumed to have no mortality below the highest age at reproduction. Fertility measures are then net

maternity measures, however there is often little difference because now, especially in more developed countries, there is little mortality below age 50. The "age structures" of the populations examined are, strictly speaking, past birth trajectories. At a later stage, if appropriate to the analysis, age-time-specific mortality can be added.

2.1 The projection relationship

The fundamental relationships to be developed allow analytical population projection over time. The population involved can be an actual/observed population or a theoretical model.

Over a single time interval, the basic projection relationship can be written

$$x_t = A_t x_{t-1} \tag{1}$$

where x_t is a column vector of population by age. In the 3-age group case, its transpose can be written (x_{1t}, x_{2t}, x_{3t}), with x_{jt} being the number of persons in age group j at time t. With 3 age groups, Leslie matrix A_t can be written

$$A_t = \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

The first row elements of A_t, the α_{jt}, are related to net maternity, and give the number of persons in the first age group at time t per person in the j-th age group at time $t–1$. The subdiagonal elements are equal to one, implementing the assumption that all persons in age groups 1 and 2 at time $t–1$ survive to be in age groups 2 and 3, respectively, at time t. It follows that, over t intervals of time, the projection relationship is

$$x_t = A_t A_{t-1} A_{t-2} \dots A_1 x_0. \tag{3}$$

2.2 The Leslie matrix in Sykes form

If there are n age groups, Leslie matrix A_t has n eigenvalues (or roots) and n right eigenvectors (Caswell 2001). Stable population theory tells us that if A_t remains constant for a sufficiently long period of time, then the dominant eigenvalue of A_t, denoted by λ_t, describes the long-term growth of the population, and the dominant right

eigenvector, $\boldsymbol{u}_t$, describes the eventual relative age composition of the population (cf. Schoen 2006).

Sykes (1973) presented a way to decompose a Leslie matrix into its dominant (stable) component and its subordinate components. To begin, consider format matrix $\boldsymbol{F}_t$ written as

$$\boldsymbol{F}_t = \begin{bmatrix} 1 - f_{2t} - f_{3t} & f_{2t} & f_{3t} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

Eq(4) shows $\boldsymbol{F}_t$ as a 3x3 matrix, but the same principles apply to any n x n matrix. Note that $\boldsymbol{F}_t$ has the structure of a Leslie matrix, i.e. nonzero elements only in the first row and subdiagonal. Further, $\boldsymbol{F}_t$ is a row stochastic matrix, meaning all of its rows sum to 1. As a result, $\boldsymbol{F}_t$ has a dominant root (eigenvalue) of 1 and a dominant right eigenvector composed entirely of ones. The elements f_{jt} represent the number of persons in age group 1 at time t per person in age group j at time $t–1$ when $\boldsymbol{F}_t$ is viewed as a projection matrix. Alternatively, the first row elements of $\boldsymbol{F}_t$ can be thought of as describing an age pattern of net maternity.

Matrix $\boldsymbol{F}_t$ can be turned into Leslie matrix $\boldsymbol{A}_t$ with dominant root λ_t and dominant right eigenvector $\boldsymbol{u}_t$ by writing the Sykes equation

$$\boldsymbol{A}_t = \lambda_t \boldsymbol{U}_{dt} \boldsymbol{F}_t \boldsymbol{U}_{dt}^{-1} = \begin{bmatrix} (1 - f_{2t} - f_{3t})\lambda_t & f_{2t}\lambda_t^2 & f_{3t}\lambda_t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5}$$

where $\boldsymbol{U}_{dt}$ is a diagonal matrix whose diagonal elements are the elements of dominant right eigenvector $\boldsymbol{u}_t$. With no mortality below the highest reproductive age, the diagonal elements of an n x n $\boldsymbol{U}_{dt}$ are $(1, \lambda_t^{-1}, \lambda_t^{-2}, \dots, \lambda_t^{-(n-1)})$, i.e. simple functions of λ_t alone. To reverse the process, that is to go from $\boldsymbol{A}_t$ to $\boldsymbol{F}_t$, first find the dominant root and right eigenvector of $\boldsymbol{A}_t$ by conventional means, and then extract them from $\boldsymbol{A}_t$. The Sykes decomposition can be verified by noting that all rows of $\boldsymbol{F}_t$ sum to 1 (cf. Schoen 2006; p139).

Sykes form separates the dominant, long-term component from all of the subordinate components of the Leslie matrix. With no mortality below the highest age of reproduction, this dominant component is a function of root λ_t alone. Matrix $\boldsymbol{F}_t$, which embodies all of the subordinate components, does little more than influence the age pattern of net maternity.

2.3 Expressing a population in terms of changes in the number of births

With no mortality before the end of reproduction, the population vector is a reflection of the trajectory of births (i.e. persons in the first age group). Let us define G_t, the proportional increase in births from time t–1 to time t, by

$$G_t = x_{1t}/x_{1,t-1}. \tag{6}$$

In many instances, population size can be arbitrarily scaled. If the size of the population is scaled so that $x_{1,t-1} = 1$, then $G_t = x_{1t}$ and the time t population can be written

$$\boldsymbol{x_t} = G_t \begin{bmatrix} 1 \\ 1/G_{t-1} \\ 1/(G_{t-1}G_{t-2}) \end{bmatrix} = \boldsymbol{g_t} . \tag{7}$$

Using Eqs (4) and (7), the projection relationship in Eq (1) can be written in Sykes form as

$$\boldsymbol{g_t} = \lambda_t \boldsymbol{U_{dt}} \boldsymbol{F_t} \boldsymbol{U_{dt}^{-1}} \boldsymbol{g_{t-1}}. \tag{8}$$

Aside from the first row elements in $\boldsymbol{F_t}$, Eq (8) expresses the projection relationship solely in terms of rates of increase, i.e. the λ_t, and G_t values. Those are the core functions of the Intrinsic Linkage approach to analytical population projection.

3. The Intrinsic Linkage relationship

In the long term, under stability, G_t becomes constant and equal to λ_t. In the short term, it is reasonable to assume that λ_t and the previous increase in births, G_{t-1}, interact to produce G_t. Let this interaction be linear. Then at time t there must be some scalar weight, w_t, such that

$$G_t = \lambda_t (1 - w_t) + G_{t-1} w_t. \tag{9}$$

Eq(9) is the Intrinsic Linkage relationship, so named because it links the intrinsic growth rate of the Leslie matrix to beginning of interval (time t–1) and end of interval (time t) proportional increases in births. Intrinsic Linkage Eq(9) is valid because any number (here G_t) can be expressed as a linear combination of any 2 other given numbers (here λ_t and G_{t-1}). For example, let λ_t and G_{t-1} be 5 and 10, respectively.

Eq(9) gives us the result 7 when weight w_t is 0.4; 12 when the weight is 1.4; and 2 when the weight is –0.6. Formally, one can find w_t from the population and Leslie matrix values (i.e. from G_{t-1}, λ_t, and G_t). Therefore, Eq(9) can apply to any population, and w_t can be seen as a parameter that relates λ_t and G_{t-1} to G_t.

3.1 Analytical projection under Intrinsic Linkage

The goal is to find the birth trajectory analytically, without the need to project the population interval by interval. Intrinsic Linkage makes an analytical population projection possible given the sequences of Leslie matrices (or the λ_t) and the w_t. This is because Eq(9) can be <u>cumulated over time</u> to express the G_t sequence in terms of the past sequences of λ_t and w_t and the time 0 (or initial) value G_0. Using straightforward algebra in Eq(9) yields

$$G_t = \left[G_0 \prod_{j=1}^{t} w_j \right] + \sum_{j=1}^{t} \lambda_j (1 - w_j) \left[\prod_{i=j+1}^{t} w_i \right]. \tag{10}$$

Eq(10) is a generally valid expression that relates the time t population's proportional increase in births to the w_t weights, initial value G_0, and the past sequence of Leslie matrix growth rates. Population sizes follow from Eqs (6) and (7) and the scaling employed. Note that the $(1 - w)$ term in Eq(10) satisfies the algebraic identity

$$1 - w = 1/[1 + w + w^2 + w^3 + \cdots]. \tag{11}$$

In a demographically realistic model, G_t must be finite and positive, so the product of the w_t must be bounded at all times. As is customary, assume the population projection matrices are non-negative and primitive. Then weak ergodicity applies (Schoen 2006, Chap. 2), and the initial population structure is eventually forgotten. Accordingly, the first (G_0) term on the right of Eq(10) must go to zero as t becomes large. As the product of the w_j becomes increasingly small, the sum in the second term on the right of Eq(10) converges.

Because Eq(9) can be seen as a tautology, Eq(10) applies to any population. This means that if one mechanically projects a given starting population using a known sequence of Leslie matrices, one can always generate the G_t from the projection, and the w_t from Eq(9) or (10). Eq(9) is of analytical value only when the w_j are known or can be assumed or estimated, as is generally the case in the rest of this paper.

3.2 Analytical projection with parameter *w* constant: Strong Intrinsic Linkage

If parameter w is *constant over time*, Eq(10) simplifies. In the constant w case, which can be termed Strong Intrinsic Linkage, Eq(10) becomes

$$G_t = [G_0 w^t] + (1 - w) \sum_{j=1}^{t} \lambda_j w^{t-j}. \tag{12}$$

Eq(12) is not based on a tautology, as specifying the w_t sequence imposes a restriction on the F_t, i.e. on the subordinate components of the sequence of Leslie matrices. That is because the given w_t require the sequence of Leslie matrices to generate the G_t sequence specified by Eq(12) [or more generally, by Eq(10)]. Since the dominant components of the Leslie matrices are determined by the given λ_t, the subordinate components, i.e. the F_t, must be constrained so that projection by Eq(1) yields the G_t produced by Intrinsic Linkage. The consequences of the constraints on F_t for the Leslie matrix elements are explored in Section 6 below.

As *t* becomes large, the G_0 term on the right of Eq(12) disappears as the effect of initial conditions on the proportional increase in births is forgotten. The Strong Intrinsic Linkage relationship then becomes

$$G_t = (1 - w) \sum_{j=1}^{t} \lambda_j w^{t-j} = (1 - w)[\lambda_t + w\lambda_{t-1} + w^2\lambda_{t-2} + w^3\lambda_{t-3} + \cdots + w^{t-1}\lambda_1]$$
$$= \frac{[\lambda_t + w\lambda_{t-1} + w^2\lambda_{t-2} + w^3\lambda_{t-3} + \cdots + w^{t-1}\lambda_1]}{[1 + w + w^2 + w^3 + \cdots + w^{t-1}]} \tag{13}$$

where the last equality follows from Eq(11) and the assumption of large t.

Eq (13) shows that, at large t, the proportional increase in the number of births between times $t-1$ and t (or the relative size of the number of persons in the first age group to that in the second at time *t*) is the sum of a convergent power series in w, where values of $|w| < 1$ to increasingly higher powers are applied to earlier period Leslie matrix growth rates. Hence function G_j is a weighted average of λ_j values, where the most recent λ_j have the greatest weight.

Eq (13) is a new result, and is significant as it provides for the analytical projection of a dynamic population based on an intuitively plausible relationship between the birth trajectory and Leslie intrinsic growth rates. Moreover, the power series in Eq(13) indicates how earlier λ_j get increasingly smaller weights, and thus become "forgotten" over time. Parameter w is a measure of convergence or of forgetting the past, as the closer w is to zero, the smaller the weight given to earlier λ_j. Strong Intrinsic Linkage can show the implications of any sequence of λ_j for the associated G_j sequence because

it constrains the subordinate components of the Leslie matrices so that past growth effects are always decaying exponentially over time.

If there is a regularity in the λ_j sequence that allows for algebraic summation, Eq(13) can provide a closed form relationship between λ_j and G_j values. <u>Stability</u> is a special case of this. If the intrinsic growth rate λ is constant, then Eq(13) yields $\lambda = G_t$ and the population is stable as parameter w drops out of the equation. The following sections examine other regularities in λ_t, including explicit functional forms and cyclicity, and consider the constraints on Leslie matrix values.

4. Functional relationships between λ_t and G_t under Strong Intrinsic Linkage

To appreciate some implications of Eq(13), we examine several functional relationships between Leslie matrix roots and birth trajectories. Specifically, we consider cases where w is constant over time and the λ_j sequence is linear, quadratic, and metastable.

4.1 The case of Leslie matrices with linearly changing roots

Linear change in population growth is quite plausible, though not often considered. Let us rewrite Eq(13) as a difference in λ_t values. This yields

$$G_t = \lambda_t + \sum_{j=1}^{t-1}(\lambda_{t-j} - \lambda_{t-j+1})w^j. \tag{14}$$

Now let the λ_j sequence change linearly, i.e. let

$$\lambda_t = b_0 + b_1 t \tag{15}$$

for constants b_0 and b_1. Then, using Eq(11), Eq(14) becomes

$$G_t = \lambda_t - b_1 w/(1-w). \tag{16}$$

Thus, when w is constant, G_t changes linearly with λ_t, with the 2 measures separated by a constant amount at every time point.

To see how the proportional increase in births, G_t, varies with parameter w in the linear case, we can take the partial derivative of Eq(16) and write

$$\partial\, G_t/\partial w = -b_1/(1-w)^2. \tag{17}$$

If $b_1 > 0$, i.e. when G_t is increasing, then an increase in w implies a larger reduction in G_t, i.e. a greater gap between G_t and λ_t.

4.2 The case of Leslie matrices with quadratically changing roots

Let λ_t change quadratically, i.e. let

$$\lambda_t = b_0 + b_1 t + b_2 t^2. \tag{18}$$

Then G_t can be found using Eqs (11), (13), and (14) and the Maple derived summation relationship (cf. Char et al 1992)

$$S = \sum_{j=1}^{t} b_2 j w^{t-j} = [b_2 t(1-w) - wb_2]/(1-w)^2. \tag{19}$$

The result is

$$G_t = \lambda_t - wb_1/(1-w) - wb_2[2t(1-w)(1+w)]/(1-w)^2. \tag{20}$$

Here, the difference between λ_t and G_t changes linearly with time.

The G_t associated with cubic and higher powers of change in λ_t can be found using the above approach, but at the price of more complex relationships. Still, at least in principle, Eq(13) can yield an explicit relationship for G_t for any polynomial λ_t.

4.3 The case of Leslie matrices from a metastable population model

The metastable model generalizes the stable model by allowing net maternity to steadily increase or decrease over time. The different formulation of Eq(14) leads to an explicit solution for G_t in the metastable case, that is where

$$\lambda_t = ck^t \tag{21}$$

and c and k are constants. When $k = 1$, the population is stable. It follows from Eq(13) that

$$G_t = ck^t[k(1-w)/(k-w)] \tag{22}$$

or that G_t is ck^t times a constant factor. This factor, previously denoted λ_s by Schoen (2006:134), depends on w and on the exponential growth parameter k. The constant w Intrinsic Linkage formulation thus accommodates metastability and provides a new, closed form expression for λ_s, specifically

$$\lambda_s = k(1 - w)/(k - w). \tag{23}$$

In Eq(21), the change in λ_t can be seen as an exponentiated linear change (i.e. as the linear change in Eq(15) exponentiated). Both the linear and the exponentiated linear patterns of change are demographically plausible and deserving of analysis.

To consider how the metastable G_t varies with parameter w, we differentiate Eq(22) with respect to w and find

$$\partial G_t/\partial w = -ck^{t+1}(k - 1)/(k - w)^2. \tag{24}$$

If $k > 1$ so that G_t is increasing over time, then an increase in w implies a smaller G_t. As in the linear case with a positive slope, a change in w moves G_t in the opposite direction.

5. Cyclically stable populations under Intrinsic Linkage

Cyclically stable populations arise naturally in a number of applications, for example when seasonality is a factor. The analysis of cyclical stability was pioneered by Skellam (1967), with significant work by Namboodiri (1969) and Tuljapurkar (1985; 1990). The Intrinsic Linkage approach is well suited to analyzing cyclical populations, and Eqs (10) and (13) yield new and explicit solutions for the G_t trajectory when λ_t varies cyclically.

5.1 The general case where cycle length is 2 intervals

Consider the simplest case under Eq(10), where λ_t alternates in value between λ_1 (when t is odd) and λ_2 (when t is even). Then, with w_1 associated with λ_1 and w_2 associated with λ_2, Intrinsic Linkage Eq(9) implies

$$\begin{aligned} G_1 &= \lambda_1(1 - w_1) + G_2 w_1 \\ G_2 &= \lambda_2(1 - w_2) + G_1 w_2. \end{aligned} \tag{25}$$

Using Eq(10) with t large, and summing the terms in λ_1 and λ_2 separately, yields the solutions

$$G_1 = [\lambda_1(1 - w_1) + \lambda_2 w_1(1 - w_2)]/[1 - w_1 w_2]$$
$$G_2 = [\lambda_2(1 - w_2) + \lambda_1 w_2(1 - w_1)]/[1 - w_1 w_2]. \tag{26}$$

Under Eq(13), with w constant, the solutions in Eq(26) are reduced to

$$G_1 = [\lambda_1 + \lambda_2 w]/[1 + w]$$
$$G_2 = [\lambda_2 + \lambda_1 w]/[1 + w] \tag{27}$$

with $G_1 + G_2 = \lambda_1 + \lambda_2$.

5.2 The case where cycle length is 3 and w is constant

When each cycle spans 3 time intervals, the 3 Strong Intrinsic Linkage specifying equations are

$$G_1 = \lambda_1(1 - w) + G_3 w$$
$$G_2 = \lambda_2(1 - w) + G_1 w$$
$$G_3 = \lambda_3(1 - w) + G_2 w \tag{28}$$

The solutions for the proportional increases in births are then

$$G_1 = (\lambda_1 + w\lambda_3 + w^2\lambda_2)/(1 + w + w^2)$$
$$G_2 = (\lambda_2 + w\lambda_1 + w^2\lambda_3)/(1 + w + w^2)$$
$$G_3 = (\lambda_3 + w\lambda_2 + w^2\lambda_1)/(1 + w + w^2) \tag{29}$$

which implies $G_1 + G_2 + G_3 = \lambda_1 + \lambda_2 + \lambda_3$.

5.3 Strong Intrinsic Linkage where cycle length is m

The approach in Section 5.2 readily generalizes to cycle lengths of m intervals, where m is any positive integer. With w constant, the m specifying equations are

$$G_1 = \lambda_1(1 - w) + G_m w$$
$$G_2 = \lambda_2(1 - w) + G_1 w$$

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$$G_m = \lambda_m(1-w) + G_{m-1}w. \tag{30}$$

Note in the first equation that G_m precedes G_1. The straightforward solution can be written

$$G_j = \sum_{i=1}^{m} w^{i-1}\lambda_{j-i+1}/\left(\sum_{i=1}^{m} w^{i-1}\right) \tag{31}$$

where the value of λ_0 is taken to be λ_m [as in Eqs(25)-(27)]. As before, $\sum G_j = \sum \lambda_j$, with the sums over j ranging from l to m.

Eq(31) shows the new Strong Intrinsic Linkage solutions for the birth trajectory in cyclically stable models of any cycle length. Under constant w, closed form solutions that are essentially truncated versions of Eq(13) link the proportional increase in the number of births to the intrinsic growth rates of each cycle's Leslie matrices.

6. Specifying the complete population projection matrices

The Intrinsic Linkage model is not complete until the underlying sequence of Leslie matrices is fully specified. With parameter w fixed, matrix $\boldsymbol{F_t}$, a constituent of Leslie matrix $\boldsymbol{A_t}$ per Eq(4), is constrained to produce the exponential decay in the effect of past behavior shown in Eq(13). This constraint may preclude the existence of a demographically valid Leslie matrix.

The best way to approach the specification of the $\boldsymbol{A_t}$ is to begin with models that have only 2 reproductive age groups.

6.1 Specifying the 2-age group Intrinsic Linkage Leslie matrix

Using Eq(4), we can write the 2 x 2 Intrinsic Linkage Leslie matrix in the form

$$A_t = \begin{bmatrix} \lambda_t(1-a_t) & \lambda_t^2 a_t \\ 1 & 0 \end{bmatrix} \tag{32}$$

where a_t is the contribution to the number of births at time t per person in the second age group at time $t-1$ when the dominant eigenvalue is 1. Schoen (2006: 138-39) shows that such a Sykes form representation is always possible, and describes how it can be

implemented. Since a_t and λ_t must be greater than zero, and we need $(1-a_t) > 0$, we must have

$$0 < a_t < 1 \qquad (33)$$

for a demographically valid population projection matrix.

Let us scale the model so that the population at time $t-1$ is described by the vector

$$\boldsymbol{x_{t-1}} = \begin{bmatrix} 1 \\ 1/G_{t-1} \end{bmatrix} \qquad (34)$$

and the time t model population is given by

$$\boldsymbol{x_t} = G_t \begin{bmatrix} 1 \\ 1/G_t \end{bmatrix}. \qquad (35)$$

Eqs(32) - (35) must (and do) satisfy the matrix projection relationship of Eqs(1) and (8).

Using those equations and the Intrinsic Linkage relationship of Eq(9), we can get a scalar equation from the first row of matrix Eq(1) and write

$$G_t = \lambda_t (1 - a_t) + a_t \lambda_t^2 / G_{t-1} = \lambda_t (1 - w_t) + G_{t-1} w_t. \qquad (36)$$

Eliminating G_t by using the last equality, we find

$$a_t = (-w_t)\, G_{t-1} / \lambda_t. \qquad (37)$$

Eq(37) provides the 2-age group Intrinsic Linkage solution for a_t, indicating that all of the elements of $\boldsymbol{A_t}$ are fully determined.

Because a_t, G_{t-1} and λ_t are always positive, Eq(37) indicates that a demographically valid model for time t requires $w_t < 0$. If w_t is 0, then a_t is 0 and $\boldsymbol{A_t}$ is not primitive. Since we must have $a_t < 1$, Eq(37) implies the inequality

$$\lambda_t > (-w_t) G_{t-1} \qquad (38)$$

for a valid Leslie matrix. The growth rate of $\boldsymbol{A_t}$ is likely to be somewhat similar in magnitude to the proportional increase in births from time $t-2$ to time $t-1$. Since $0 < (-w_t) < 1$, the magnitude of w_t may be constrained to satisfy the inequality in Eq(38). A smaller $(-w_t)$ implies more rapid convergence and a lesser effect of past behavior.

Combining Eqs(37) and (38), a valid time t model must have

$$0 > w_t > -\lambda_t/G_{t-1}. \tag{39}$$

For a valid 2-age model, either w_t or λ_t is constrained by Eq(39). With a time-varying w_t, however, it is possible that at some time z, w_z can be smaller than –1.

<u>Under Strong Intrinsic Linkage</u>, with w constant, $-1 < w < 0$ is needed in order to have convergence in Eqs(12) and (13). Using Eq(13), we can write the inequality in Eq(38) as

$$\lambda_t > (-w)(1-w)[\lambda_{t-1} + w\lambda_{t-2} + w^2\lambda_{t-3} + w^3\lambda_{t-4} + \cdots]. \tag{40}$$

Eq(40) provides a necessary and sufficient condition for a valid constant w, 2-age group Leslie matrix. From Eq(9), the Strong Intrinsic Linkage relationship implies that λ_t is always between G_t and G_{t-1}.

With constant w and a known λ_t sequence, closed form solutions for a_t in terms of w and the λ_t may be possible. For example, consider the 2-age group metastable model of Eqs(21) - (23). Eq(37) leads to

$$a = (-w)(1-w)/(k-w) \tag{41}$$

with parameter a constant over time. Since in most cases the metastable parameter k is close to 1, $a \approx -w$ and the demographic validity of the model can quickly be established.

In the 2 age group, cycle length 2 case, Leslie matrix parameters a_1 and a_2 are given by

$$\begin{aligned} a_1 &= [-w/(1+w)](\lambda_2 + \lambda_1 w)/\lambda_1 \\ a_2 &= [-w/(1+w)](\lambda_1 + \lambda_2 w)/\lambda_2. \end{aligned} \tag{42}$$

A valid model thus requires a_1 and $a_2 < 1$, as well as $(\lambda_2 + \lambda_1 w) > 0$ and $(\lambda_1 + \lambda_2 w) > 0$.

6.2 Specifying the 3-age group Intrinsic Linkage Leslie matrix

Leslie matrices with three reproductive age groups can provide a reasonable representation of most human population dynamics. Let the first row of Sykes form 3-age group Leslie matrix $\boldsymbol{A_t}$ be $[(1 - a_t - b_t)\lambda_t, a_t\lambda_t^2, b_t\lambda_t^3]$, with the matrix having ones on the subdiagonal and zeros elsewhere. We must also have $0 \leq a_t,\ b_t \leq 1$ and $(1 - a_t - b_t) \geq 0$. The time $t{-}1$ model population can be scaled so that

$$\boldsymbol{x}'_{t-1} = [1, 1/G_{t-1}, 1/(G_{t-1} G_{t-2})] \tag{43}$$

where the prime (') indicates the transposition from a column vector to a row vector.

Eq(9) and the first row of matrix projection Eq(1) then yield the equation

$$(-w)/\lambda_t = a_t/G_{t-1} + b_t[(\lambda_t^2 - G_{t-1}G_{t-2})/(G_{t-1}G_{t-2}\{\lambda_t - G_{t-1}\})]. \tag{44}$$

With three age groups there are 2 parameters in $\boldsymbol{F_t}$, a_t and b_t, which are constrained by only one equation, i.e. Eq(44). Thus a valid Leslie matrix may arise in many ways, and a_t and b_t are not fully determined by the Intrinsic Linkage constraint. In the general case, this flexibility makes it possible to have a value of w_t that is less than −1 or greater than 1, as long as the sum in Eq(10) always converges.

Three points should be made. <u>First</u>, if $b_t = 0$, then we again have the case of Eq(37). Hence Eq(39) (or Eq(40) if w is constant) provides a sufficient condition for a valid model, though it is no longer a necessary condition.

<u>Second</u>, if w is constant over time, a long-term valid Intrinsic Linkage model again requires that $-1 < w < 0$. To see why, consider the model at time *t*, and assume that G_{t-1} and G_{t-2} are known. If $G_{t-1}=G_{t-2}$, then the population is stable (or $\boldsymbol{A_{t-1}}$ is not primitive). If not, let us assume that $G_{t-2} > G_{t-1}$ (with the same line of argument holding if $G_{t-2} < G_{t-1}$). From Eq(44), for $w \geq 0$, a valid model requires that the second term on the right be negative, hence λ_t must be between G_{t-1} and G_{t-2} and greater than the geometric mean of G_{t-1} and G_{t-2}. With $w \geq 0$, Eq(9) implies that G_t must equal λ_t or be between λ_t and G_{t-1}. Thus at the beginning of the next interval, time *t+1*, we are back at the starting situation, but the gap between G_{t-1} and G_t is substantially smaller than the gap that existed between G_{t-1} and G_{t-2}. Over time, with $w \geq 0$, the gap between successive G's must go to zero, and the population will either become stable or the population projection matrix will no longer be primitive. Since $w \geq 0$ does not yield a valid long-term model, we are left with $-1 < w < 0$.

<u>Third</u>, because the 3-age group model has an additional degree of freedom, we can introduce another constraint, say one related to the Net Reproduction Rate (NRR). Following Lotka, the NRR can be thought of as the growth that occurs over a generation, as in the stable population relationship

$$NRR = \exp(rT) \tag{45}$$

where r is Lotka's intrinsic growth rate and T is the length of a generation (Schoen 2006:11). A 3-age group model typically has the reproductive age groups 0-14, 15-29, and 30-44, and age 30 is often close to the length of a generation. In the discrete case, with λ_t representing 15 years of growth from time *t−1* to time *t*, we can thus write

$$NRR_t = \lambda_t^2 = (1 - a_t - b_t)\lambda_t + a_t\lambda_t^2 + b_t\lambda_t^3. \tag{46}$$

Using Eqs(46), (1), and (9), we find that the parameters of $\boldsymbol{A_t}$ are given by

$$a_t = \{(-w)G_{t-1}G_{t-2}(\lambda_t + 1)[G_{t-1} - \lambda_t] - \lambda_t G_{t-1}G_{t-2}\}/DENOM$$

and

$$b_t = \{G_{t-2}[wG_{t-1} + \lambda_t][G_{t-1} - \lambda_t]\}/DENOM$$

where
$$DENOM = \lambda_t^2[G_{t-2}(G_{t-1} - 1) - \lambda_t(G_{t-2} - 1)]. \tag{47}$$

An analyst can start with a fixed w and a sequence of NRRs, and generate sequences of λ_t's, G_t's, and Leslie matrices.

6.3 Specifying the Intrinsic Linkage Leslie matrix for 4 or more age groups

Leslie matrices with four or more age groups have relationships that parallel those in 3-age models. Eqs(39) and (40) still provide sufficient conditions for a valid Leslie matrix in the general and constant w cases, respectively. In equations that parallel Eq(44), there are additional terms on the right side, increasing the likelihood that a valid Leslie matrix exists. For example, in the 4-age case, there is an additional term of the form

$$[\lambda_t^3 - G_{t-1}G_{t-2}G_{t-3}]/[G_{t-1}G_{t-2}G_{t-3}(\lambda_t - G_{t-1})]. \tag{48}$$

Still, in the constant w case, the same dynamic discussed in the second point after Eq(44) continues to operate, so in long term the value of w must satisfy $-1 < w < 0$.

7. Numerical illustrations involving Intrinsic Linkages

Numerical values for Intrinsic Linkage models are presented for (i) a time series of observed data for Netherlands, Sweden, and the United States; (ii) a metastable model, and (iii) a 3-cycle cyclically stationary model. Models with three reproductive age groups are used to simplify the presentation while preserving the essential population dynamics.

7.1 Intrinsic Linkage values in data for Netherlands, Sweden, and the United States

We examine the Intrinsic Linkage parameters underlying observed demographic behavior in three countries for which consistent data are available for a number of consecutive years. Keyfitz and Flieger (1968; 1990) provide such series for Netherlands, Sweden, and the United States. Through 1965, the data are in 5-year age groups through ages 80-85, as well as an 85 and over age group; after 1965, ages 0 and 1-4 are recognized separately. Using published data for births and the intrinsic growth rate, values of λ_t and G_t were calculated, and Eq(9) was then used to find the implied value of w_t. This value of w_t is quite insensitive to the number of age groups in the data.

Table 1 shows the λ_t, G_t, and w_t values for those three countries. The data G_t series show a number of fluctuations, with values ranging from below 75% to over 130%. The values in the λ_t series are smoother, varying only from 94.5% to 111%. In contrast, those implied in the w_t series are extremely erratic. Few values are between zero and –1, and most are positive. There are a number of double digit w_t values, ranging from –15 to +29. The constant w assumption is thus a strong one, in that parameter w appears to be far from constant in actual populations. Nonetheless, a constant w does characterize metastable models, which are demographically plausible. Empirically, the constant w assumption leads to reasonable demographic values and trajectories. Strong Intrinsic Linkage is thus likely to be appropriate for hypothetical and illustrative analyses. In addition, Intrinsic Linkage models can relax or forego the constant w assumption. Parameter w can be allowed to vary within each cycle of a cyclically stable population, change according to some functional form, or track some arbitrary sequence.

The specific value of a constant parameter w seems to be of relatively little importance for the resultant G_t sequence. The last two columns of Table 1 show that in all three countries there are only modest differences in the G_t when w changes from –0.2 to –0.8. When the G_t are cumulated over time (not shown), the constant w values are generally close to each other and to the cumulated λ_t values, and are quite different from the cumulated data G_t.

Table 1: **Observed and proportional changes in births and Intrinsic Linkage parameter values, for Netherlands, 1908-12 to 1985; Sweden, 1783-88 to 1985; and United States 1924-26 to 1985**

Year (t)	Observed Increase in Births G_t	Implied w_t [Eq(9)]	λ_t	G_t from Eq(9), w= −0.2	G_t from Eq(9), w= −0.8
		A. Netherlands, 1908-12 to 1985			
1908-12	0.992		1.071	0.992	0.992
1913-17	1.018	0.650	1.065	1.080	1.124
1918-22	1.038	0.439	1.054	1.049	0.999
1923-27	1.003	2.848	1.057	1.059	1.104
1928-32	0.995	1.210	1.043	1.039	0.994
1933-37	0.957	2.261	1.025	1.022	1.050
1938-42	1.070	−0.592	1.028	1.029	1.011
1943-47	1.300	29.363	1.062	1.069	1.103
1948-52	0.987	−0.330	1.064	1.063	1.033
1953-57	0.981	1.076	1.061	1.060	1.083
1958-62	1.050	0.220	1.070	1.072	1.059
1965	1.013	3.627	1.065	1.063	1.069
1970	0.974	2.560	1.038	1.033	1.013
1975	0.745	−15.155	0.960	0.946	0.918
1980	1.019	−0.315	0.953	0.955	0.982
1985	0.983	0.503	0.945	0.944	0.916
		B. Sweden, 1783-87 to 1985			
1783-87	0.935		1.007	0.935	0.935
1788-92	1.071	−0.564	1.022	1.092	1.039
1793-97	1.138	−0.958	1.039	1.122	1.059
1798-1802	0.950	−0.614	1.022	0.942	1.014
1803-07	1.030	0.032	1.032	1.106	1.036
1808-12	1.013	0.366	1.004	0.922	0.997
1813-17	1.056	−0.819	1.037	1.128	1.045
1818-22	1.091	3.583	1.043	0.974	1.042
1823-27	1.082	0.594	1.070	1.147	1.075
1828-32	0.979	−1.951	1.047	0.968	1.042
1833-37	1.048	0.178	1.063	1.139	1.067
1838-42	0.973	26.270	1.051	0.981	1.048
1843-47	1.062	−0.134	1.052	1.108	1.052
1848-52	1.075	2.322	1.052	1.007	1.052
1853-57	1.067	0.771	1.038	1.062	1.035
1858-62	1.122	7.831	1.059	1.056	1.063
1863-67	1.028	−0.498	1.059	1.062	1.058
1868-72	0.905	8.739	1.044	1.029	1.041
1873-77	1.112	−0.356	1.058	1.081	1.062
1878-82	0.999	−1.025	1.057	1.037	1.056
1883-87	1.021	0.652	1.061	1.080	1.062
1888-92	0.968	2.385	1.059	1.042	1.058
1893-97	0.997	0.690	1.059	1.074	1.060
1898-1902	1.030	0.459	1.059	1.047	1.059
1903-07	0.989	2.497	1.058	1.067	1.058
1908-12	1.003	0.782	1.055	1.045	1.054
1913-17	0.920	3.921	1.032	1.021	1.027
1918-22	0.986	0.315	1.017	1.013	1.015
1923-27	0.858	17.270	0.994	0.979	0.990
1928-32	0.881	0.786	0.966	0.956	0.962
1933-37	0.935	0.236	0.951	0.947	0.949

Table 1: **(continued)**

Year (t)	Observed Increase in Births G_t	Implied w_t [Eq(9)]	λ_t	G_t from Eq(9), w= −0.2	G_t from Eq(9), w= −0.8
1938-42	1.150	−4.394	0.975	0.997	0.980
1943-47	1.312	2.284	1.024	1.047	1.033
1948-52	0.888	−0.420	1.014	0.987	1.010
1953-57	0.921	0.731	1.010	1.029	1.010
1958-62	0.975	0.388	1.009	0.994	1.009
1965	1.171	−2.836	1.026	1.052	1.030
1970	0.897	−0.477	0.986	0.932	0.977
1975	0.941	0.402	0.970	1.001	0.969
1980	0.937	1.203	0.962	0.930	0.960
1985	1.014	−1.520	0.967	0.997	0.969
C. United States 1924-26 to 1985					
1924-26	0.991		1.046	0.991	0.991
1929-31	0.923	4.098	1.013	1.030	1.017
1934-36	0.944	0.700	0.994	0.965	0.990
1939-41	1.104	−1.639	1.005	1.036	1.008
1944-46	1.235	2.907	1.035	1.034	1.040
1949-51	1.232	0.980	1.074	1.107	1.081
1954-56	1.125	0.165	1.105	1.103	1.109
1959-61	1.044	−4.772	1.111	1.118	1.112
1965	0.893	8.257	1.065	1.023	1.056
1970	0.982	0.353	1.030	1.036	1.025
1975	0.843	−10.716	0.970	0.917	0.959
1980	1.149	−1.320	0.975	1.021	0.998
1985	1.016	0.231	0.976	0.941	0.976

Note: United States data, up to and including 1965, is from the "adjusted births" series. The proportional increase in births, G_t, is calculated from data on births at times t and t−1 using Eq(6). Values of λ_t are calculated as exp[5r_t], using published values for female intrinsic growth rate r_t.
Source: Keyfitz and Flieger (1968) for data through 1965; Keyfitz and Flieger (1990) for data from 1970 through 1985.

We have yet to consider the implications of the choice of constant *w* for Leslie matrix elements. However, using the criterion in Eq(38) as a rule of thumb, it appears that for –0.8 ≤ *w* ≤ –0.2, demographically valid Leslie matrices can be found for every country and year shown but one. In Sweden 1943-47, G_t was 1.312, while λ_t for 1938-42 was only 0.975, less that 80% of G_t. For the Swedish series, a constant *w* would have to be smaller in magnitude than 0.74 in order to consistently yield valid population projection matrices.

7.2 Values in a 3-age group metastable model

The metastable model allows fertility to steadily increase or decrease, and it can provide an analytical bridge between stable population regimes, such as in a transition from stability to stationarity. Table 2 shows numerical relationships under two values of the constant Intrinsic Linkage parameter *w*, specifically −0.6 and −0.4. Values for the

metastable parameters are c=1.02 and k=1.005, with the Leslie matrix value b set at 0.05.

Table 2: **Numerical relationships in a 3-reproductive age group birth-death metastable model with Intrinsic Linkage, for 2 values of constant parameter w**

Item	w = −0.6	w = −0.4
a	0.41747	0.22790
λ_s	1.00187	1.00142
λ_1	1.02510	1.02510
G_1	1.02702	1.02656
λ_{10}	1.07216	1.07216
G_{10}	1.07417	1.07369
First row of Leslie matrix A_1	[.54589, .43869, .05386]	[.74023, .23948, .05386]
First row of Leslie matrix A_{10}	[.57095, .47990, .06162]	[.77421, .26198, .06162]
Time 0 model population x'_0	[1, .97856, .96237]	[1, .97900, .96323]
Time 10 model population x'_{10}	(1.63397) [1, .93561, .87974]	(1.62672) [1, .93603, .88052]

Note: The model is based on text Eqs(21)-(23) with c=1.02, k=1.005, and b=0.05. Population growth over time was accumulated using the relationship that the product of k^j, j going from 1 to n, equals $k^{n(n+1)/2}$. The value of parameter a in the 3-age metastable model is found from $a = \{bk(k-w)^3 - (1-w)^2[w(1-k) + b(k-w)]\}/\{w^2(1-k) - k^2(1-w) + (k-w)\}$.

Because the metastable population is growing, fertility values in the first row of the Leslie matrix increase over time, while the relative size of the population in the older age groups decreases. Leslie growth rate λ_t increases over time, as does $G_t = \lambda_t \lambda_s$. When *w* increases from −0.6 to −0.4, there is a slight decrease in metastable growth factor λ_s, but over a ten-interval period the resultant difference in births is only that between 1.634 and 1.627, a mere 0.4%. However, the change in *w* does have a considerable effect on age-specific fertility values, with the larger *w* associated with an earlier fertility schedule. Because the value of b was fixed, the first rows of both Leslie matrices have identical third elements at the same time point.

7.3 Values in a 3-age group, 3-cycle cyclically stationary model

A cyclically stationary population provides analysts with a flexible model for studying fluctuations in the absence of long-term growth. The Intrinsic Linkage approach provides explicit relationships that facilitate the construction of such models.

Table 3 shows values for a 3-cycle, cyclically stationary, Intrinsic Linkage model with three age groups. For stationarity, we must have G_1 G_2 G_3 = 1, i.e. no growth over

each cycle. Combined with Eqs(29), the stationarity constraint enables the model to be solved directly for the G_t and constant w, though the λ_t must be chosen carefully to produce a valid model. An easier approach is to choose a value of w in the broad range that satisfies the sufficiency condition in Eqs(39) or (40), choose two of the three λ's, and solve for the G_t and the last λ. Table 3 does the latter, with $\lambda_1 = 0.9$, $\lambda_2 = 1.1$, and w set at either −0.8 or −0.4. In the Leslie matrix, there is only one constraint on each pair of a_j and b_j. To simplify matters while insuring a valid model, b_j is always set at 0.02.

Table 3: **Numerical relationships in a Cyclically Stationary 3-reproductive age group, 3-cycle birth-death model with Intrinsic Linkage, for 2 values of constant parameter w**

Item		w = −0.8	w = −0.4
	λ_3	1.03749	1.02517
	G_1	0.92144	0.87623
	G_2	1.24285	1.18951
	G_3	0.87320	0.95944
	a_1	.94148	0.33271
	a_2	.61814	0.28422
	a_3	.95108	0.46533
	b_1	.02	.02
	b_2	.02	.02
	b_3	.02	.02
First row of Leslie matrix A_1		[.03467, .76260, .01458]	[.58256, .26949, .01458]
First row of Leslie matrix A_2		[.39805, .74794, .02662]	[.76535, .34391, .02662]
First row of Leslie matrix A_3		[.03001, 1.02373, .02233]	[.52763, .48905, .02155]
Time 0 model population x'_0		[1, 1.14521, .92144]	[1, 1.04228, .87623]
Time 1 model population x'_1		(.92144)[1, 1.08526, 1.24285]	(.87623)[1, 1.14126, 1.18951]
Time 2 model population x'_2		(1.14521)[1, .80460, .87320]	(1.04228) [1, .84068, .95944]

Note: The model is based on text Eqs(28)-(29) with λ_1=0.9 and λ_2=1.1. Since $G_1\,G_2\,G_3$=1, there is no population growth over each 3-interval cycle.

Table 3 indicates that the choice of w has a small effect on the calculated value of λ_3, but a clear impact on the values of the G_t. The more negative value of w leads to large values for the a_j, with a resultant concentration of fertility in the second age group. The less negative value of w is associated with a marked shift in fertility to the first age group. Because the cycle length is the same as the number of age groups, total population size is constant over time, and equals 3.0666 when w=−0.8, and it is 2.9185 when w=−0.4. Population vector x_3 is identical to x_0, and Leslie matrix A_4 is the same as A_1. Within a cycle, however, cohort size varies by up to 24% when w=−0.8 and up to

19% when $w=-0.4$. For both choices of w, the sum of the three G_t equals the sum of the three λ_t.

8. Summary and Conclusions

The Intrinsic Linkage approach of Eq(9) is based on the idea that it is plausible to represent proportional increase in births G_t as a linear combination of λ_t and G_{t-1}. That allows each G_t to be expressed by a convergent power series in λ_t and linear weight parameter, w_t.

Intrinsic Linkage emphasizes the dominant, stable component of Leslie matrices. Much of the demographic content of Leslie matrix $\boldsymbol{A_t}$, of whatever size, is embodied in a single number: dominant eigenvalue λ_t. The dominant root of $\boldsymbol{A_t}$ reflects not only intrinsic growth but, in the assumed absence of mortality before the end of childbearing, the long-term age composition as well. The contribution of the subordinate components of $\boldsymbol{A_t}$ is captured by Intrinsic Linkage parameter *w*. The analytical advantage of the Intrinsic Linkage approach flows from its ability to focus on just λ and *w*. As presented here, Intrinsic Linkage yields the birth trajectory, but mortality assumptions (or data) can be added to provide the complete population age structure.

When λ_t has a polynomial, exponential, or cyclical trajectory and *w* is constant over time, new closed form relationships between λ_t and G_t emerge. Over the long term, Eq(13) shows that G_t is a weighted average of present and past λ_t values, with past effects decaying exponentially over time. The essence of the constant *w* or "Strong" Intrinsic Linkage approach is that it imposes a strict regularity on how past behavior is forgotten. By doing so, it extends the concept of a force of convergence (Schoen 2006; Chap. 3)

The construction of Intrinsically Linked models is straightforward. Several illustrative numerical examples of cyclical and metastable models are given, and these demonstrate the validity of the new theoretical relationships derived here. The constant *w* assumption, when applied to data, produces a birth trajectory that, while quite reasonable, resembles the λ_t sequence more than the actual birth sequence. While the value of *w* has only a modest influence on the birth sequence, it substantially affects the age pattern of net maternity.

In observed populations, however, the weight parameter is generally <u>not</u> constant over time. The time series data suggest that past effects are typically forgotten in a complex, irregular fashion. Still, stable populations have great analytical and applied value even though most actual populations are not stable in their age composition nor are they consistent in their underlying intrinsic growth rates. The Intrinsic Linkage approach is rooted in population dynamics, and addresses the crucial relationship

between the implicit level of stable growth and the explicit nature of year-to-year growth by formalizing the manner in which past experiences are forgotten.

In sum, the Intrinsic Linkage approach provides a new and flexible way to analytically project a birth trajectory. Intrinsic Linkage extends current methods for the dynamic modeling of birth-death models, including cyclically stable and cyclically stationary models, and affords new options for analyzing populations with changing vital rates.

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