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Formal Relationships 33

Outsurvival as a measure of the inequality of lifespans between two populations

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Outsurvival as a measure of the inequality of lifespans between two populations

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Abstract

BACKGROUND
Inequality in lifespans between two populations, e.g., males and females or people with low and high socioeconomic status, is a focus of demographic, economic, and sociological research and of public policy analysis. Such inequality is usually measured by differences in life expectancy.

OBJECTIVE
We aim to devise a cogent measure of how much distributions of lifespans differ between two populations.

RESULTS
We propose an *outsurvival statistic*, φ (phi), that measures the probability that an individual from a population with low life expectancy will live longer than an individual from a population with high life expectancy. This statistic can also be interpreted as an underdog probability – the chance that a random value from a distribution with a low mean will exceed a random value from a distribution with a higher mean.

CONCLUSIONS
Our outsurvival probability complements life-expectancy differences to provide a more nuanced view of the inequality of lifespans between two populations. Our mathematically

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equivalent underdog probability provides an intuitive and widely applicable perspective on the more general question of how disparate two distributions are.

1. Relationship

Consider two populations. Suppose ages at death (lifespans) in the first population are independent of ages at death in the second population. Let $\mu_i(x), i = 1, 2$, denote the force of mortality at age x in two populations, which could be males and females or people with low and high socioeconomic status (SES). Assume that the first population has the lower life expectancy. Let $\ell_i(x)$ denote survival over age and let $d_i(x) = \mu_i(x)\ell_i(x)$, denote the density of ages at death (i.e., lifespans). Consider two randomly selected individuals or, equivalently, all possible pairs of individuals, one from each population, at the starting age, which could be birth or, say, age 65. The probability that the individual from the first population will outlive the individual from the second population is:

$$\varphi = \int_0^\infty d_2(x)\ell_1(x)dx = \int_0^\infty \mu_2(x)\ell_2(x)\ell_1(x)dx. \tag{1}$$

Proof of (1) is straightforward. In the first variant of the formula, the value of $d_2(x)$ gives the probability that the lifespan of an individual from population 2 is x and $\ell_1(x)$ is the probability that the lifespan of an individual from population 1 is greater than x. Hence the product is the chance that the individual from the first population will outsurvive the individual from the second population. Summing the products over age gives the overall probability for all individuals.

As described later, there are various measures of the difference between two distributions. Use of Equation (1) to shed light on lifespan inequality between two populations is, to our knowledge, new. We developed the measure φ rather than relying on some other approach because of our interest in how often individuals in one group live longer than individuals in another group. Hence, Equation (1) pertains to the distribution of lifespans in two populations. A lifespan has a start (e.g., birth or attaining some age or the start of use of some equipment such as an automobile or a light bulb) and an end (e.g., death or the failure of the equipment). Usually lifespans are measured by the duration between birth and death in years or some other unit of time. For an automobile, an alternative measure of duration could be kilometers driven; for a rabbit, number of offspring; for a worker, cumulative wages; for a scholar, articles published or citations received.

2. Related results

Note that Equation (1) implies that the chance an individual from the second population will outlive a person from the first population is $\int_0^\infty \mu_1(x)\ell_1(x)\ell_2(x)dx$. Since this probability is $1-\varphi$, it follows that

$$(2) \qquad 1-\varphi = \int_0^\infty \mu_1(x)\ell_1(x)\ell_2(x)dx$$

and

$$(3) \qquad \int_0^\infty \big(\mu_1(x)+\mu_2(x)\big)\ell_1(x)\ell_2(x)dx = 1.$$

If the two populations have the same age patterns of mortality or, equivalently, if two individuals share the same $\mu(x)$, $\ell(x)$, and $d(x)$, then (1) and (3) lead to

$$(4) \qquad \varphi = \int_0^\infty d(x)\ell(x)dx = \int_0^\infty \mu(x)\ell^2(x)dx = 0.5.$$

The probability φ is zero if the longest lifespan in the first population is less than the shortest lifespan in the second population. The value of 0.5 is not the maximum possible value of φ. Consider an extreme case when everyone in the first population dies close to the same age, X_1, so life expectancy for this population is X_1. Suppose for the second population, 99% of the population dies at birth, but the remaining 1% survive to an age $X_2 > 100X_1$. The life expectancy of the second population, $0.01X_2$ exceeds X_1, but φ is 0.99. As a second example, suppose that for 99% of the first population the age at death is close to a value X_1 but 1% die at birth. Life expectancy is $0.99X_1$. In contrast, suppose for the second population that everyone has a lifespan close to X_2 but less than X_1 with $0.99X_1 < X_2 < X_1$. Again φ is 0.99. More generally, φ is a probability such that $0 \leq \varphi < 1$.

Suppose the two populations have constant forces of mortality over age, $\mu_1(x) = \mu$ and $\mu_2(x) = \mu/2$. Then life expectancy for the first population, given by $1/\mu$, is half the life expectancy of the second population. Because

$$(5) \qquad \varphi = \int_0^\infty \mu_2 e^{-\mu_2 x} e^{-\mu_1 x} dx = \frac{\mu_2}{\mu_1+\mu_2},$$

the value of φ is $1/3$. Despite the large difference in life expectancy, an individual in the first population has one chance in three of outliving a contemporary in the second population.

Instead of pertaining to two individuals from two populations, Equation (1) can be used to analyze the risk of dying from some cause. Then φ gives the probability of death from a cause, called cause 2, rather than any other cause, grouped together as cause 1, with the underlying assumption that causes 1 and 2 are independent. The equation makes it clear that to die from cause 2, a person has to survive death from cause 1 (i.e., from all hazards except cause 2). This is well known but sometimes forgotten. The chance that a person from some population will outlive a person from another population is conceptually different from the chance a person dies from some cause rather than another cause, but φ is an appropriate measure in both cases.

Equation (1) is readily generalized to discrete distributions (Bergeron-Boucher et al. 2020).

3. Applications

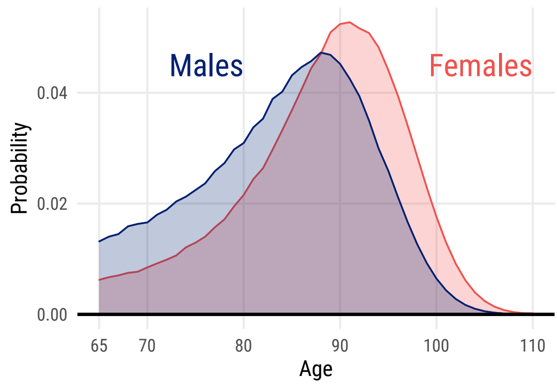
Stratification has stimulated considerable interest: How distinct are two populations based on the distribution of some characteristic capturing, such as social status or political or religious views (Shi et al. 2021; Zhou and Wodtke 2019)? Figure 1 provides an illustration that uses lifespan data. The distributions were estimated from life tables for French males and females in 2018; these life tables were renormalized to start with age 65. One measure of the stratification of the two distributions is the overlapping index (Pastore 2018; Gini and Livada 1943), the area coloured purple, which is given by

(6) $$\eta = \int_0^\infty min\big(d_1(x), d_1(x)\big)dx.$$

In Figure 1, the overlapping area η is 0.81. Because the area under the density distribution of deaths is 1, the combined area of the non-overlapping distributions, coloured red and blue, is $2(1-\eta)$.The red area is actually the same as the blue area, $(1-\eta)$. The proportion of the non-overlapping areas to the total area (i.e., the ratio of the sum of the areas coloured red and blue to the sum of those two areas and the common area coloured purple), is $2(1-\eta)/(2-\eta)$ and has a value of 0.32 in Figure 1; this statistic was used by Shi et al. (2021).

Our approach, as given by the first formula in Equation (1), is illustrated in Figure 2, using the same French data used in Figure 1. Figure 2a displays $\ell_1(x)$, the survival curve for the French males. Figure 2b shows $d_2(x)$, the distribution of ages at death for the French females, the population with higher life expectancy. The area shaded in purple in Figure 2b is the area defined by the product of the $\ell_1(x)$ and $d_2(x)$ curves: This area, which equals φ, is 0.37. Hence, although remaining life expectancy at age 65 was 19.47 for French males, almost four years less than the value of 23.39 for French females, 37% of the males – about three in eight –would outlive a corresponding female.[4] Note that because death is inevitable, the area under the d_2 curve is 1. Hence the grey area in Figure 2(b) is 1 – 0.37, or 0.63.

Figure 1: Life table age at death distributions from age 65 for French males and females, 2018. The ratio of the blue plus red areas to the total area is 32%



Note: Males are population 1 and females are population 2.
Source: Source: Own calculations applied to the Human Mortality Database (2020) French life tables by sex for 2018.

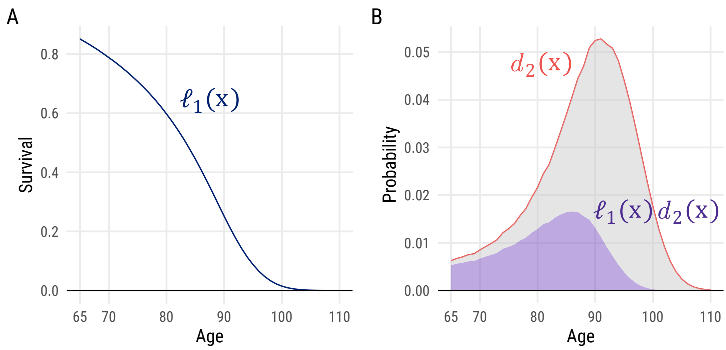
Life tables for the United States in 2017 (Center for Disease Control and Prevention 2019) show that non-Hispanic blacks had a life expectancy at birth of 74.9 years compared with 78.5 years for non-Hispanic whites. Yet φ was 0.439: In nearly 44% of possible pairs, the non-Hispanic black would outlive the non-Hispanic white.

Life tables by education level in Italy for the year 2012 (ISTAT 2020) show that, at age 30, men with a university degree had a remaining life expectancy of 53.1 years, while men with lower secondary education had a remaining life expectancy of 50.2 years. Our computations indicate that φ was 0.431.

Substantial differences in life expectancy often lead to values of φ greater than 40% (Bergeron-Boucher et al. 2020). This surprised the authors of this article and many of the researchers with whom we discussed drafts. Even population experts may not have an intuitive feel for how much two distributions of lifespans overlap.

[4] The value of φ was estimated using the discrete approximation defined in Bergeron-Boucher et al. (2020) applied to the Human Mortality Database (2020) life tables by sex. Two other ways to estimate φ were analyzed by Bergeron-Boucher et al. (2020): (1) random pairing of individual lifespans, based on simulated individual lifespans from empirical death rates, and (2) continuous approximation using Equation (1) based on simulated lifespan distributions. All three methods provided equivalent results.

Figure 2: Graphical representation of φ: A – Survival curve of population 1, French males; B – life table age at death distribution of population 2, French females and the product of female age at death distribution and male survival (purple area)



Note: The value of φ is given by the purple area; grey area represents the complementary value $1 - \varphi$.
Source: Own calculations using the discrete approximations defined in Bergeron-Boucher et al. (2020) applied to the Human Mortality Database (2020) French life tables by sex for 2018.

4. Directions for further research

Our measure φ of between-population lifespan inequality could be applied widely. For instance, Bergeron-Boucher et al. (2020) have recently completed an analysis of how male-female lifespan inequality differs across populations and is changing over time. A similar study could be done of inequality among SES groups in how long people live after retirement. The measure could also be applied to nonhuman and nonliving populations, for example, to study how often female birds in some species outlive corresponding male birds or how often some automobile make and model is still on the road compared with another make and model (Vaupel et al. 1998).

How does life expectancy vary with perturbations in mortality? This is a question that has led to some informative mathematical relationships (Vaupel 1986; Wrycza and Baudisch 2012). A similar question can be asked: How does outsurvival, φ, vary with perturbations in mortality? In particular, how much would lifespan inequality between two populations be reduced or increased by different patterns of change in age-specific mortality and resulting life expectancy? As a simple example, consider Equation 5, which implies

$$\varphi = \frac{1}{1 + \frac{\mu_1}{\mu_2}} = \frac{1}{1 + \frac{e_2}{e_1}}, \tag{7}$$

where life expectancy e_i is the inverse of the constant force of mortality μ_i. If the ratio of life expectancies is 2, then outsurvival is a third, but if the ratio is 3, then φ worsens to 0.25.

The specific value of φ depends on the difference between the life expectancies of the two population and also on the shape of the distributions. It would be interesting to derive some formal relationships.

Also interesting for future research would be to focus on the cases when an individual from the first population outlives an individual from the second population to work out outsurvival time (i.e., how much longer the first individual lives). Suppose φ is 0.4 and suppose that the fortunate 40% outlive the average individual in the advantaged population by x months. Whether x is 1 month or 60 months may be important.

Calculations of φ assume that the two lifespan distributions are independent. This is often not true for matched individuals, such as spouses or twins. More generally, a complicated problem that merits further study is how joint probability density functions should be estimated for correlated lifespans of pairs of individuals who may not be the same age. The math has to be worked out, but Equations (1–4) suggest that the more similar the hazard functions and survival functions are, the higher φ will tend to be.

5. Extension

Here we consider extensions from the case of two lifespan distributions to the more general case of two probability functions. Let $f_1(x)$ and $f_2(x)$ be two probability density functions and let $F_1(x)$ and $F_2(x)$ be the corresponding cumulative distributions. Then,

$$\varphi = \int_0^\infty f_1(x)F_2(x)dx. \tag{8}$$

This corresponds to Equation (1) with $f_1(x) = d_1(x)$ and $F_2(x) = \int_0^x d_2(t)dt$ because

$$\begin{aligned}
\varphi &= \int_0^\infty d_2(x)\ell_1(x)dx \\
&= \int_0^\infty d_2(x)\int_x^\infty d_1(t)dtdx \\
&= \int_0^\infty d_1(x)\int_0^x d_2(t)dtdx \\
&= \int_0^\infty f_1(x)F_2(x)dx.
\end{aligned} \tag{9}$$

Note that in the derivation, Equation (9), there is a switch in the order of integration between line 2 and line 3. Equation (1) and (8) are equivalent when studying lifespan inequalities. Bergeron-Boucher et al. (2020) also show this equivalency and how Equation (1) relates to the joint probability density function.

If in Equation (8), the first density distribution has a lower mean than the second density distribution, then φ could be called the underdog probability – the chance that a value from the distribution with the lower mean turns out to be bigger or greater than the value from the corresponding distribution with the higher mean. Calculation of underdog probabilities can be applied to a wide range of topics. For example, it can be used to study income inequality between sexes. According to the US Census Bureau, the mean income for females was \$43,315, while it was \$65,144 for males in 2019 (United States Census Bureau 2020). Applying Equation (8) to the income distributions, we can estimate that the probability of females having a higher salary than males is 38.5%.

As a second example, consider the ages of people in Italy and Ireland. In 2019 the median age in Italy was 46.2 whereas the median age in Ireland was 37.2 (Eurostat 2020). If a random person from Ireland was matched with a random Italian, in two out of five pairs the Irish person would be older.

A related research question was recently studied in bibliometrics (Brito and Rodríguez-Navarro 2019). The authors challenged the commonly used practice of evaluating an article based on the impact factor of the journal in which it is published. In the experiment, randomly chosen articles from a pair of journals were compared based on their journals' impact factors and the actual citations gained in the recent years. Via extensive random paring the researchers estimated the proportion of cases when the article in the journal with the higher impact factor was the article that was more cited. This turned out to be close to 50%, like flipping a coin, indicating that the common strategy of proxying an academic paper's quality with the impact factor of the journal is problematic.

6. Discussion

How much two density distributions differ is a topic of a vast literature that includes overlapping indices as well as calculations of within and between group variance. One major line of research is the development and application of approaches to determine how large the vertical difference is between two distributions according to measures of statistical distance: Two examples are Kullback-Leibler divergence and Kolmogorov distance. These and various measures of horizontal differences (the earth mover's problem) have been applied to distributions of lifespans, income, IQ, temperature, and other topics. (D'ambrosio 2001; Deza and Deza 2016; Edwards and Tuljapurkar 2005; Naveau, Guillou, and Rietsch 2014; Permanyer and Scholl 2019).

Our measure φ is a probability that sheds light on a different question: How distinct are two density distributions? If the mean of one distribution is bigger than the mean of the other, it is sometimes claimed this holds for all individuals. For example, 'the life expectancy of men is shorter than that of women' is often sloppily summarized as 'men do not live as long as women'. Let X_1 be the remaining lifespan of a man and let X_2 be the remaining lifespan of a woman of the same age. Then the probability that $X_1 > X_2$ is φ. More generally, φ captures the probabilistic correctness of the assertion that individuals from a distribution with a low mean have values less than those of individuals from a distribution with a higher mean. Probabilistic correctness is a different concept than within vs. between group variance, the overlap of two density distributions, or the statistical distance according to some metric between two distributions.

Disparities between groups raise social concerns and fuel political debate. Consider, for example, discrepancies in lifespans after age 65 for people who started full-time work as teenagers or later. Suppose, to keep the example simple, that people who started work before age 20 have a remaining life expectancy at age 65 that is five years less than for people who started work after age 20. Should the people who started work early get bigger monthy retirement benefits? If φ = 0.4, then 40% of the people given higher benefits would wind up living longer than people with lower benefits. It might be argued that such 'misallocation' would 'inappropriately benefit' 40% of the people who started work early. There would be less concern if φ were 0.1. Hence, outsurvival probabilities may be of some use in guiding ways to define and restrict membership in groups to stratify a population into distinct groups. The life expectancy difference between two groups is also important. Beyond summary statistics such as φ and life expectancy gaps, distributions (such as shown in Figures 1 and 2) should be analyzed.

In some contexts, values of φ and corresponding life expectancy gaps may be highly correlated. In a companion article by Bergeron-Boucher et al. (2020), data from the Human Mortality Database (2020) are studied for 44 countries and regions and for all years available, with a focus on differences between female and male lifespans. The gap between female and male life expectancy was strongly associated with φ: r^2 was –0.90.

Nonetheless, outsurvival probabilities provide useful information. For instance, when φ was around 0.3, female life expectancy was usually about 10 years higher than male life expectancy – but sometimes the gap was 20 years. And when the gap was 10 years, sometimes 40% of males outlived females.

Equations (1) and (8) are not difficult to prove and have been used in various contexts when it is of interest to estimate the chance that a random variable from one distribution exceeds a random variable from another distribution. In some contexts, Equation (8) is known as the probability of superiority (e.g., Ruscio, 2008). As a graduate student in 1969, the first author of this Formal Relationship studied Equation (8) when he was learning about game theory: If two opponents have different uncertain strengths, what is the chance the first will win? Another example comes from reliability theory: What is the risk that some uncertain stress on some equipment will exceed the uncertain strength of the equipment (An, Huang, and Liu 2008)?

In short, the contribution made here is not the proof of a new equation. The contribution is the innovative interpretation of straightforward mathematics to open a new vista on comparing two demographic distributions.

7. Replicability

All data and R code to replicate the results have been submitted together with the article and can be found in an additional online file and in the GitHub repository https://github.com/CPop-SDU/outsurvival-demres, also versioned with Zenodo https://doi.org/10.5281/zenodo.4573914.

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