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Formal Relationships 6

Survival as a function of life expectancy

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Guest Editors are Joshua R. Goldstein and James W. Vaupel.

Table of Contents

Survival as a function of life expectancy

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Abstract

It is well known that life expectancy can be expressed as an integral of the survival curve. The reverse - that the survival function can be expressed as an integral of life expectancy - is also true.

1. Relationship

Let $e(a)$ denote remaining life expectancy at age a and let $\ell(a)$ denote the proportion surviving to age a (the survival function). Then

(1) $$l(a) = \frac{e(0)}{e(a)} \exp\left(-\int_0^a \frac{1}{e(x)}\, dx\right).$$

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2. Proof

The population composition of a life table population, i.e., the probability density function (pdf), is given by

(2) $$c(a) = \ell(a)/e(0).$$

The proportion of the population age a and older is given by

(3) $$C(a) = \int_a^\infty c(x)\,dx.$$

The intensity of change in C with age is given by the relative derivative

(4) $$\gamma(a) = \frac{\frac{dC(a)}{da}}{C(a)} = \frac{-c(a)}{C(a)} = \frac{-\ell(a)/e(0)}{\int_a^\infty \ell(x)\,dx/e(0)} = \frac{-1}{\int_a^\infty \ell(x)\,dx/\ell(a)} = -\frac{1}{e(a)}.$$

Since $C(a) = C(0)\exp\left(\int_0^a \gamma(x)\,dx\right)$ and $C(0) = 1$, equation (4) implies that

(5) $$C(a) = \exp\left(-\int_0^a \frac{1}{e(x)}\,dx\right).$$

It follows from (2) and (3), as shown in (4), that

(6) $$C(a) = e(a)\ell(a)/e(0).$$

Substituting (6) in (5) and rearranging terms yields (1).
Q.E.D.

A slightly more compact proof can be based on $T(a) = \int_a^\infty \ell(x)\,dx = e(a)\,\ell(a)$ instead of equation (3).

3. History and related results

This elegant proof based on the pdf in (2) is due to Meilijson (1972). Finkelstein (2008) provides related information about it. The pdf in (2), which is well known to demographers, plays an important role in different branches of probability and statistics. For instance, it describes the duration of the first cycle in $[0, \infty]$ for a renewal process that started at $t = -\infty$ (Ross 1996). It also provides asymptotic (as $t \to \infty$) expressions for the distributions of backward and forward recurrent times for renewal processes. This result has an important demographic interpretation for stationary populations, namely, that the distribution of age in a life table population is equal to the distribution of remaining lifetimes (Goldstein 2009; Vaupel 2009).

Because

$$\ell(a) = \int_a^\infty d(x)\,dx = \int_a^\infty \mu(x)\ell(x)\,dx, \tag{7}$$

where $d(x)$ is the pdf of deaths and $\mu(x)$ denotes the force of mortality, it follows from (4) that

$$\frac{1}{e(a)} = \frac{\ell(a)}{\int_a^\infty \ell(x)\,dx} = \frac{\int_a^\infty \mu(x)\,\ell(x)\,dx}{\int_a^\infty \ell(x)\,dx} = \bar{\mu}(a), \tag{8}$$

where $\bar{\mu}(a)$ is the weighted average of the force of mortality above age a. Because $\bar{\mu}(a)$ is the usual deaths/person-years measure for an open interval, equation (8) has a nice intuitive interpretation: among those surviving to age a, there will eventually be one death for every $e(a)$ person-years lived and therefore the average force of mortality above age a must be $1/e(a)$.

4. Applications

As discussed by Gupta and Akman (1995), Mi (1995), Gupta and Kirmani (2000), and Finkelstein (2002), relationships derived from (1) can be used to analyze how the shape of the age-trajectory of mortality and the shape of the age-trajectory of remaining life expectancy influence each other. We provide here a single illustration, not in these articles, of a demographic application.

Taking the relative derivative of both sides of (1) and re-arranging terms leads to

$$\frac{\frac{de(a)}{da}}{e(a)} \equiv \acute{e}(a) = \mu(a) - \frac{1}{e(a)} \tag{9a}$$

and, using (8), to

$$\acute{e}(a) = \mu(a) - \bar{\mu}(a). \tag{9b}$$

Instead of deriving these formulas from (1), they can be derived from the following important result:

(10)
$$\frac{de(a)}{da} = \mu(a)\,e(a) - 1.$$

This result is easily proven by differentiating both sides of $e(a) = \int_a^\infty \ell(x)\,dx/\ell(a)$. Hence, instead of deriving (9a) from (1), an alternative proof of (1) can be based on integrating both sides of (9a) derived from (10).

In any case, (9a) indicates that remaining life expectancy will rise, stay constant or fall depending on whether the force of mortality exceeds, equals, or is less than the inverse of remaining life expectancy. Similarly, (9b) implies that remaining life expectancy will rise, stay constant or fall depending on whether the force of mortality exceeds, equals, or is less than the average force of mortality after age a. It might be naively thought that remaining life expectancy always falls when the force of mortality rises, or that remaining life expectancy always rises when the force of mortality falls, but (9a) and (9b) show that this is not necessarily the case. What is critical in determining whether remaining life expectancy rises or falls with age is not whether the force of mortality is rising or falling but whether the force of mortality exceeds or is less than the inverse of remaining life expectancy.

If the force of mortality is ultimately monotone after some age a_m , this observation can be made more explicit. Suppose $\mu(a)$ is monotonically increasing for $a \geq a_m \geq 0$. Then its weighted average in $[a_m, \infty)$ is obviously larger than $\mu(a_m)$ and in accordance with (9b), $\acute{e}(a)$ is monotonically decreasing.

In many human populations as well as populations of various non-human species and some kinds of equipment, the force of mortality falls from birth to a minimum at some age (for humans and other mammals often around the age of reproductive maturity) and then rises afterwards. Equation (9a) indicates that remaining life expectancy will rise after birth if and only if the force of mortality at age 0 exceeds the inverse of life expectancy at age 0. If it does, and if the force of mortality decreases until some age and then increases, remaining life expectancy will initially increase and eventually start to decline. (See Theorem 2.6 in Finkelstein (2008:35) for the formal proof of a general result). Consequently, there will be an age at which remaining life expectancy reaches a maximum. This age, (9a) implies, is the age when the force of mortality equals the inverse of remaining life expectancy. In general, it will not be the age when the force of mortality reaches a minimum.

The Swedish female life table for 1751 available at www.mortality.com provides central death rates at various ages. They can be used as approximations to the force of mortality. At age 0, the central death rate is given as 0.2122, and is much greater than the inverse of life expectancy at birth, 1/39.88=0.0251. Remaining life expectancy rises at ages 1, 2, 3 and 4 and reaches a maximum at age 5 of 50.44, with an inverse value of 0.0198. At age 4 the central death rate is $0.0237 > 0.0199$, the inverse of remaining life expectancy at age 4, whereas at age 5, the central death rate is $0.0188 < 0.0198$, the inverse of remaining life expectancy at age 5. This is consistent with remaining life expectancy reaching a maximum around age 5.

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