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Formal Relationship 22

Variance in age at death equals average squared remaining life expectancy at death

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Tomasz F. Wrycza [1]

Abstract

BACKGROUND
Variance in life span σ^2 and life expectancy lost due to death $e^\dagger$ are important demographic indicators of life disparity.

OBJECTIVE
I show that the variance in age at death equals the average squared remaining life expectancy at death. Based on this finding, I also show that the average squared difference in remaining life expectancy at death equals the difference between σ^2 and ${e^\dagger}^2$.

COMMENTS
Calculations of some of the quantities involved for the Gompertz-Makeham mortality model with varying parameters produce complex patterns.

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1. Relationship

Let

$$e(x) = \frac{1}{l(x)} \int_x^\infty l(t)dt$$

denote remaining life expectancy at age x. If $f(x)$ denotes the probability density function of age at death,

$$e^\dagger = \int_0^\infty e(x)f(x)dx$$

denotes the average number of life-years lost due to death and

(1) $$\sigma^2 = \int_0^\infty (x - e_0)^2 f(x)dx$$

denotes the variance in age at death, then it holds that:

(2) $$\sigma^2 = \int_0^\infty e^2(x)f(x)dx,$$

i.e., the variance in age at death is the average squared remaining life expectancy at death, and

(3) $$\frac{1}{2}\int_0^\infty \int_0^\infty (e(x) - e(y))^2 f(x)f(y)dx\,dy = \sigma^2 - {e^\dagger}^2,$$

i.e., the average squared difference in remaining life expectancy at death equals the difference between the variance and the square of life expectancy lost due to death.

2. Proof

(2): Applying integration by parts with $u(x) = e^2(x)$ and $v'(x) = f(x)$ gives

$$\int_0^\infty e^2(x)f(x)dx = e_0^2 + 2\int_0^\infty e(x)e'(x)l(x)dx.$$

Since $e'(x) = e(x)\mu(x) - 1$, it follows that

$$\int_0^\infty e^2(x)f(x)dx = e_0^2 + 2\int_0^\infty e^2(x)f(x)dx - 2\int_0^\infty e(x)l(x)dx \quad \Rightarrow$$

$$\Rightarrow \int_0^\infty e^2(x)f(x)dx = 2\int_0^\infty e(x)l(x)dx - e_0^2.$$

According to Goldstein (2009), it holds that

$$\int_0^\infty e(x)l(x)dx = \int_0^\infty xl(x)dx.$$

Hence

$$\int_0^\infty e^2(x)f(x)dx = 2\int_0^\infty xl(x)dx - e_0^2.$$

Integration by parts shows that

$$2\int_0^\infty xl(x)dx = \int_0^\infty x^2 f(x)dx.$$

Hence

$$\int_0^\infty e^2(x)f(x)dx = \int_0^\infty x^2 f(x)dx - e_0^2 \overset{(1)}{=} \sigma^2.$$

Q.E.D.
(3): This follows from (2) since

$$\begin{aligned}\int_0^\infty \int_0^\infty (e(x)-e(y))^2 f(x)f(y)dx\,dy = \\ = 2\int_0^\infty e^2(x)f(x)dx - 2\left(\int_0^\infty e(x)f(x)dx\right)^2 = \\ \overset{(2)}{=} 2(\sigma^2 - {e^\dagger}^2).\end{aligned}$$

Q.E.D.

3. History and related results

Note that dividing (3) by e_0^2 (which means considering $e(x)/e_0$; i.e., remaining life expectancy in units of e_0) results in a relationship between two other demographically meaningful quantities, life table entropy $\bar{H}$ and the coefficient of variation c_v:

$$(4)\qquad \frac{1}{2}\int_0^\infty \int_0^\infty \left(\frac{e(x)-e(y)}{e_0}\right)^2 f(x)f(y)dx\,dy = \frac{\sigma^2 - {e^\dagger}^2}{e_0^2} = c_v^2 - \bar{H}^2.$$

Life table entropy

$$\bar{H} = \frac{e^\dagger}{e_0}$$

is the elasticity of life expectancy with respect to a proportional change in mortality. This was first derived by Leser (1955), and was restated by Keyfitz (1977a,1977b) in a continuous formulation. Mitra (1978), Goldman and Lord (1986) and Vaupel (1986) independently derived the mathematical expression for life disparity $e^\dagger$, and showed that $\bar{H} = e^\dagger/e_0$. Vaupel and Canudas-Romo (2003) showed that the derivative of life expectancy over time is given by the product of $e^\dagger$ and the rate of progress in reducing age-specific death rates. Recent research papers involving $e^\dagger$ include Zhang and Vaupel (2009), Vaupel (2010) and Vaupel, Zhang, and van Raalte (2011).

As a basic measure in statistics, standard deviation σ has been used in demography for research on levels and trends in the variance in adult life span in different countries over time (Edwards and Tuljapurkar 2005,2011). In these papers, a standard deviation of life span above age 10 is used in order to avoid the distorting effect of infant mortality. The ratio

$$c_v = \frac{\sigma}{e_0}$$

denotes the coefficient of variation and is used as a (normalized) measure of inequality in various fields (Allison 1978).

Research papers which have compared both $e^\dagger$ (or $\bar{H}$) and σ (or σ^2) as measures of uncertainty in age at death include Hakkert (1987), Hill (1993), and van Raalte and Caswell (2013). However, the relationships discussed here are new in that they establish explicit analytical expressions that show how σ^2 relates to remaining life expectancy $e(x)$ (relationship (2)), and how the difference between σ^2 and ${e^\dagger}^2$ should be interpreted (relationship (3)).

4. Applications

It is easy to show that

$$\frac{1}{2}\int_0^\infty \int_0^\infty (x-y)^2 f(x) f(y) dx\, dy = \sigma^2, \tag{5}$$

i.e., that the average squared difference in life span is the variance in age at death. Relationship (3) is the analogous result, if the focus is on remaining life span $e(x)$, rather than on life span x. It shows that in this case the average squared difference is not σ^2, but $\sigma^2 - {e^\dagger}^2$. This result can also be seen to give a new interpretation to $e^\dagger$:

$${e^\dagger}^2 = \frac{1}{2}\int_0^\infty (x-y)^2 f(x) f(y) dx\, dy - \frac{1}{2}\int_0^\infty (e(x)-e(y))^2 f(x) f(y) dx\, dy,$$

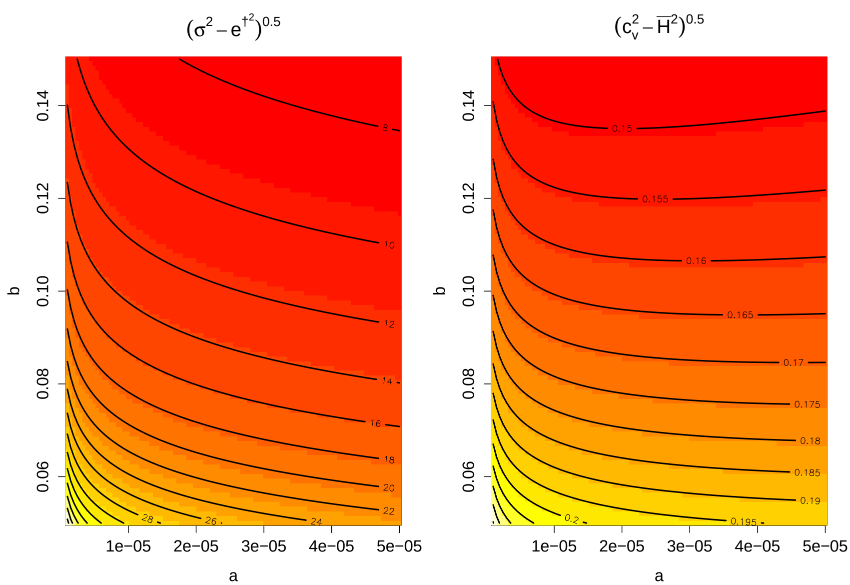
i.e., ${e^\dagger}^2$ is the difference of the average squared difference in life span minus the average squared difference in remaining life expectancy.

A trivial consequence of (3) is that σ is never smaller than $e^\dagger$ (and, accordingly, c_v is never smaller than $\bar{H}$), since the left-hand side of the equation is by definition nonnegative. Another consequence is that the only distribution for which $f(x) \neq 0\ \forall x$ and $\sigma = e^\dagger$ (or $c_v = \bar{H}$) is the exponential distribution; i.e., the distribution with constant mortality at all ages (since these assumptions, together with (3), imply that $e(x) = e(y)$ for all ages x, y). Thus the values of c_v and $\bar{H}$ can only be equal when $c_v = \bar{H} = 1$ (under the assumption $f(x) \neq 0\ \forall x$).

Relationship (3) (and the standardized version (4)) permits analysis of the dynamics of the average difference in remaining life expectancy (a double integral) by relating it to well-known measures of life span inequality (two simple integrals), and thus could be useful for future research on this topic.

For illustration of (3) and (4), assume Gompertz-Makeham mortality $\mu(x) = ae^{bx} + c$. Figure 1 shows contour plots of $\sqrt{\sigma^2 - {e^\dagger}^2}$ and $\sqrt{c_v^2 - \bar{H}^2}$ for $a \in [0.000001, 0.00005]$, $b \in [0.05, 0.15]$ and $c = 0.001$ fixed.

Figure 1: Contour plots of $\sqrt{\sigma^2 - {e^\dagger}^2}$ and $\sqrt{c_v^2 - \bar{H}^2}$



We can see that within the given range in parameters, the average difference in remaining life expectancy, as measured by

$$\sqrt{\frac{1}{2}\int_0^\infty \int_0^\infty \left(e(x)-e(y)\right)^2 f(x)f(y)dx\,dy} = \sqrt{\sigma^2 - {e^\dagger}^2},$$

is decreasing when a and/or b are increasing; the average therefore seems to track the value of e_0, which is also decreasing when a and/or b are increasing. However, when we look at the average difference in *standardized* remaining life expectancy, as measured by

$$\sqrt{\frac{1}{2}\int_0^\infty \int_0^\infty \left(\frac{e(x)-e(y)}{e_0}\right)^2 f(x)f(y)dx\,dy} = \sqrt{c_v^2 - \bar{H}^2},$$

the behavior is somewhat different. For fixed a, the value still decreases when b is increasing, but for fixed b, the pattern is hump-shaped: decreasing up to some value of a, then increasing.

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