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Formal Relationships 31

Born once, die once: Life table relationships for fertility

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Abstract

BACKGROUND
Everyone dies, and only once. This basic truth underlies all formal mortality research. Similarly, everyone is born, and only once. This basic truth has not been fully exploited to benefit formal fertility research. An advance has recently been made by Baudisch and Stott (2019), who conceive a population of unborn children awaiting the event of their own birth. This approach introduces a novel survivorship concept for birth.

RESULTS
Formalizing the idea of "birth survival," here we define the underlying random variable and derive the central triplet of survival analysis functions – hazard, density, and survival. We demonstrate that using a "born once, die once" analogy results in a straightforward framework to capture age-specific patterns of birth, analogous to classical life table functions. Based on a single variable (age-specific birth counts), we construct a "birth table" and, from there, meaningful summary measures such as "birth expectancy" and associated measures of spread.

CONTRIBUTION
We advance a new framework to enrich the toolbox of fertility research. The relationships developed here serve to compare birth schedules across populations and reveal macro-level patterns and constraints. The triplet of birth functions and the birth table set the stage to transfer methods from mortality to fertility research. They offer a starting point to study birth and death within the same framework and for the same focal individual. With analogous formal methods, studies of the intertwined relationships between birth and death become possible. This, we envision, will open an entirely unexplored line of research.

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1. Relationship

Much of formal mortality research relies on the basic observation that everyone will die once. Much of formal fertility research is limited by the uncertainty of childbearing, as women may give birth once or many times, or forego reproduction altogether. Here we demonstrate how this limitation can be resolved. For a woman, fertility is hard to foretell, but for a child, its own birth is certain, just as certain as its eventual death. A "born once, die once" paradigm allows one to formally exploit the analogy between birth and death and thereby to define a new research framework.

A classic and central framework in demographic research is survival analysis. It pervades demographic studies, not just for death but also for many other decrement processes, such as marriage, divorce, menopause, entry to and exit from the labor market, incarceration, and much more (Preston, Heuveline, and Guillot 2000). Birth as a decrement process is typically studied for a population of women who successively become mothers of different parity. Here we take an unusual view of the decrement process of birth, which recently has been suggested by Baudisch and Stott (2019). Rather than women becoming mothers, we consider the children who – conceptually – wait to be born, starting from the onset age of childbearing. Children persist in the unborn state until their event of birth. They constitute a to-be-born offspring population. This offspring population is decremented by the birth of children to mothers of certain ages. We define this offspring population retrospectively for a cohort of women with completed childbirth. This approach does not account for which specific child belongs to which specific woman. It therefore does not need to distinguish between normal, twin, or multiple births. It also does not account for a woman's survival, because survival is necessarily implied by the observed birth of her child. For such a defined offspring population, we derive the (i) survival, (ii) hazard, and (iii) probability density functions that describe waiting time to one's own birth. We follow classic survival analysis and its notation conventions.

Babies are born to women of different ages. Some babies are born to younger women; they await (survive) the event of birth for only a short time. Other babies are born to older women; they await the event of birth for a long time. Hence, we define the continuous random variable X as waiting time of a child to the event of its own birth from the onset age of the reproductive life span, α.

We define $b(x)$ as the number of babies born to women of age x, and $B(x)$ as the cumulative number of babies born to women up to age x as

$$B(x) = \int_{\alpha}^{x} b(a)\, da \tag{1}$$

over the full age range of childbearing, from a minimum age of α up to a maximum age of β, with $B(\alpha) = 0$ and $B(\beta) = B$ for an offspring population of total size B. A survival function $S(x)$ then captures the proportion of unborn children to women up to age x,

(2) $$S(x) \,=\, 1 \,-\, \frac{B(x)}{B}.$$

It holds that $S(\alpha) = 1$ and $S(\beta) = 0$.

With this definition of survivorship we derive a triplet of functions that fulfills the fundamental relationship of survival analysis between survival $S(x)$, hazard $h(x)$, and density function $f(x)$ such that

(3) $$h(x) \,\equiv\, \frac{f(x)}{S(x)} \,=\, -\frac{\frac{dS(x)}{dx}}{S(x)} \,=\, -\frac{d\ln(S(x))}{dx},$$

which implies that

(4) $$S(x) \,=\, e^{-\int_{\alpha}^{x}\, h(a)\, da}.$$

We find that the instantanous hazard $h(x)$ to be born to a woman of age x can be expressed as

(5) $$h(x) \,=\, \frac{b(x)}{B - B(x)}.$$

This function calculates the number of babies born to women of age x as a fraction of those not yet born to women up to age x. The associated probability density function $f(x)$ is given by

(6) $$f(x) \,=\, \frac{b(x)}{B}.$$

This function calculates the number of babies born to women of age x as a fraction of the total offspring population.

2. Proof

The derivative of survivorship (2) is

$$\frac{dS(x)}{dx} = -\frac{1}{B}\,\frac{dB(x)}{dx}. \tag{7}$$

By the fundamental theorem of calculus, the derivative of cumulative reproduction (1) with respect to age equals

$$\frac{dB(x)}{dx} = b(x). \tag{8}$$

With (8), the negative ratio of (7) and (2) defines the hazard function as

$$h(x) = -\frac{\frac{dS(x)}{dx}}{S(x)} = \frac{\frac{b(x)}{B}}{\frac{B-B(x)}{B}} = \frac{b(x)}{B - B(x)}, \tag{9}$$

which proves (5).

Due to the relationship (3) between survival, hazard, and density, the probability density function can be calculated from equations (2) and (5) as

$$f(x) = h(x)S(x) = \frac{b(x)}{B}, \tag{10}$$

which proves (6) and thereby completes the central triplet of birth functions.

3. Related results and extensions

The survivorship concept for birth suggested above invites one to formulate a life table concept for birth. We unravel a full "birth table," using standard life table notation (Preston, Heuveline, and Guillot 2000) for analogous and intuitive translation.

We define $\ell(x)$ as the number of children left to be born to women age x and above, which is

$$\ell(x) = \int_x^{\beta} b(a)\,da. \tag{11}$$

These "survivors" are the unborn children in the offspring population who still await the event of being born. In this context, survival means that children have not yet experienced the event of their own birth. Function (11) complements the definition of cumulative reproduction function $B(x)$ in (1) through the relationship

$$B = \ell(x) + B(x). \tag{12}$$

The number of children $B(x)$ born up to age x together with the number of children $\ell(x)$ to be born after age x add up to the total size B of the offspring population. Further, (12) implies that $B(\alpha) = \ell(\beta) = 0$ and $B(\beta) = \ell(\alpha) = B$.

Birth exposure at a focal age is given by the population of children at risk of being born to women in the age group x to $x + n$. By definition, these are the children not yet born until age x, i.e. $\ell(x)$. Birth occurrence is given by ${}_nb_x$, which denotes the children born to women in the age group x to $x + n$. Therefore, the probability of being born between ages x and $x + n$ is given by

$${}_nq_x = \frac{{}_nb_x}{\ell(x)}. \tag{13}$$

The retrospectively defined offspring population is a closed population. The only change in population size comes from births of the members of the population. Therefore we can express the number of children born within a focal age group as the difference in the number of unborn babies between successive ages, ${}_nb_x = \ell(x) - \ell(x + n)$ with $\ell(x) \geq \ell(x + n)$ for all ages $x \in [\alpha, \beta]$. Inserting into the equation above we arrive at an expression for the probability that a child is born to a woman within the focal age group:

$${}_nq_x = \frac{\ell(x) - \ell(x + n)}{l(x)} = 1 - \frac{\ell(x + n)}{\ell(x)}. \tag{14}$$

Consequently, the probability that a child is not born within the focal age group of women is given by

$${}_np_x = \frac{\ell(x + n)}{\ell(x)}. \tag{15}$$

As the cohort of women ages, each potential child either gets born or not. There is no other alternative, such that

(16)
$$ {}_nq_x + {}_np_x = 1. $$

Depending on whether a child is born or not during an age interval, it contributes an amount of person-years of exposure time to the risk of birth. Analogous to classic life table calculation, the number of person-years spent in the unborn state, ${}_nL_x$, between ages x and $x + n$ is given by the area under the survivorship curve $\ell(x)$,

(17)
$$ {}_nL_x = \int_x^{x+n} \ell(a)\, da. $$

The number of remaining person-years spent in the unborn state above age x is given by

(18)
$$ T_x = {}_{\beta-x}L_x = \int_x^{\beta} \ell(a)\, da. $$

These measures of person-years are essential to calculate key summary measures from the birth table.

First, (18) helps determine "birth expectancy" as the expected number of years to be spent in the unborn state,

(19)
$$ e(\alpha) = \frac{T_\alpha}{l(\alpha)} = \frac{\int_\alpha^\beta \ell(a)\, da}{\ell(\alpha)}, $$

with $\ell(\alpha) = B$. In other words, it captures the mean waiting time of a child to be born, starting from the onset age of the reproductive life span α. Analogously, remaining birth expectancy at any age x within the reproductive life span is given by

(20)
$$ e(x) = \frac{T_x}{\ell(x)} = \frac{\int_x^\beta \ell(a)\, da}{\ell(x)}. $$

Second, the measure of exposure, (17), is vital to calculate the birth rate, ${}_nm_x$,

(21)
$$ {}_nm_x = \frac{{}_nb_x}{{}_nL_x}. $$

The birth rate ${}_nm_x$ measures the ratio of births to potential births within the focal age range.

Note that the discrete birth table framework developed in this section converges to the continuous relationships initially proven. In the limit, the birth rate ${}_nm_x$ defines a child's age-specific hazard of birth $h(x)$ as

$$h(x) = \lim_{n\to 0} {}_nm_x = \lim_{n\to 0} \frac{{}_nb_x}{{}_nL_x} = \frac{b(x)}{\ell(x)} = \frac{b(x)}{B - B(x)}, \tag{22}$$

which is consistent with (5).

Similarly, the density and survival functions mark the limit of the discrete birth table functions as

$$f(x) = \lim_{n\to 0} \frac{{}_nb_x}{B} = \frac{b(x)}{B}, \tag{23}$$

consistent with (6), and

$$S(x) = \lim_{n\to 0} \frac{{}_nL_x}{B} = \frac{\ell(x)}{B} = \frac{B - B(x)}{B}, \tag{24}$$

consistent with (2).

The framework developed here is directly analogous to the life table and survival analysis functions in mortality research (Preston, Heuveline, and Guillot 2000). It uses standard demographic tools and invites a range of further extensions. A natural next step, for example, is to define measures of variation within our framework. Given the definitions of $e(\alpha)$ and density function $f(x)$, the variance in waiting time of an infant to be born is:

$$\sigma^2 = \int_\alpha^\beta ((a - \alpha) - e(\alpha))^2 f(a)\, da, \tag{25}$$

such that $a \geq \alpha$. From (25), the standard deviation $\sigma = \sqrt{\sigma^2}$ and coefficient of variation $CV = \frac{\sigma}{e(\alpha)}$ immediately follow. These measures describe how wide (in absolute and relative terms) a set of births is spread around their mean $e(\alpha)$. Analysis of other measures of spread as well as perturbation analysis and comparative research based on mean and (relative) spread are interesting perspectives for future research (Baudisch 2011; van Raalte and Caswell 2013; Wrycza, Missov, and Baudisch 2015; Aburto et al. 2019, 2020).

4. Applications

The code and data to reproduce the results and graphs presented in this section are publicly available through the repository https://github.com/jssalvrz/Born-Once-Die-Once.

4.1 Illustration of the relationship

The triplet of birth functions $h(x)$, $f(x)$, and $S(x)$ enables macro-level comparisons between countries. Depending on the research question, these comparisons would require deeper analysis, but for illustrative purposes, Figure 1 depicts the age pattern in the birth hazard and its associated density and survivorship functions, here illustrated for the 1960 cohort in Denmark, Germany, and the United States.

The hazard of birth increases steeply upon the onset of reproductive ages. It dampens soon thereafter but continues to rise throughout the reproductive age range. At the last ages, birth for the remaining few children over the remaining small age window implies irregularities at the level of the highest hazard, which in the limit would approach infinity.

The density function follows a hump-shaped pattern of human age-specific reproduction but differs from the age-specific maternity function conceptually. Rather than the number of children per mother, it captures the percentage of children born over the age range.

Survival falls steeply throughout and levels off at later reproductive ages as the percentage of babies left to be born approaches zero.

Together, these functions offer a condensed view of the process of age-specific reproduction, which can be analyzed by algebraic operations that are directly analogous to the analysis of mortality patterns.

Figure 1: Hazard, density, and survivorship functions of birth for Denmark, Germany, and the United States, Cohort 1960

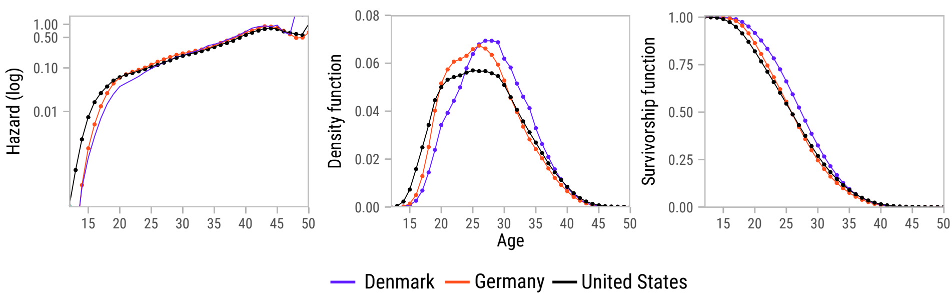


Table 1 shows an example of a birth table calculated for the Danish birth cohort of women born in 1960. The first column defines the age of a mother at the birth of a child. The following columns are directly analogous to standard life table analysis (Preston, Heuveline, and Guillot 2000): l_x, children left to be born; ${}_nb_x$, number of children born; ${}_nq_x$, probability of being born; ${}_np_x$, probability of not being born; ${}_nL_x$, person-years spent in the unborn state; T_x, remaining person-years above age x; e_x, birth expectancy; and ${}_nm_x$, age-specific birth rate.

Figure 2 exemplifies age-specific patterns of ${}_nq_x$ and e_x over age for Denmark, Germany, and the United States. The probability of being born rises steadily up to a certain age (around age 40–50, depending on the population), then it slightly declines. By the end of the reproductive life span, the last remaining child has to be born and ${}_nq_x$ equals 1. Birth expectancy, on the other hand, goes down with age. The bulge at older ages might be related to the decrease in the probability of being born, as is visible in the left panel. The source of such a pattern is a question for further research.

Figure 2: Age-specific probability of being born and remaining birth expectancy at age *x*, Denmark, Germany, and the United States, Cohort 1960

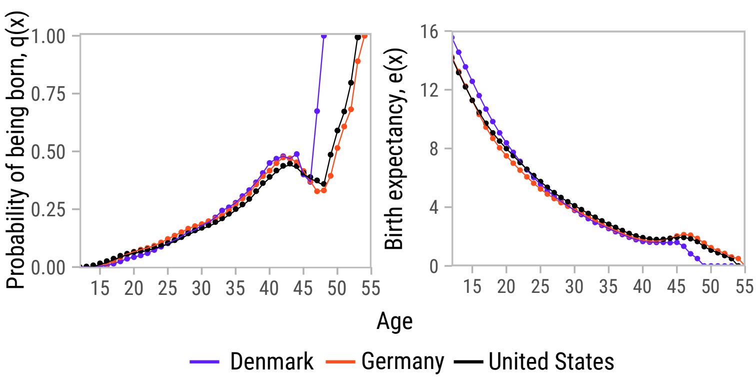


Table 1: Birth table pertaining to Denmark, Cohort 1960

Age	Number of children left to be born at age x	Children born between ages x and $x+n$	Probability that a child is born between ages x and $x+n$	Probability that a child is not born between ages x and $x+n$	Person-years spent in the unborn state between ages x and $x+n$ [1]	Remaining person-years above age x	Birth expectancy	Birth rate
x	$l(x)$	$_nb_x$	$_nq_x = \frac{_nb_x}{l(x)}$	$_np_x$	$_nL_x$	T_x	$e(x)$	$_nm_x = \frac{_nb_x}{_nL_x}$
12	70,590	1	$\frac{1}{70,590}$	$1 - \frac{1}{70,590}$	70,590	1,098,241	15.56	$\frac{1}{70,590}$
13	70,589	13	$\frac{13}{70,589}$	$1 - \frac{13}{70,589}$	70,583	1,027,652	14.56	$\frac{13}{70,583}$
14	70,577	61	$\frac{61}{70,577}$	$1 - \frac{61}{70,577}$	70,546	957,069	13.56	$\frac{61}{70,546}$
15	70,516	186	$\frac{186}{70,516}$	$1 - \frac{186}{70,516}$	70,423	886,523	12.57	$\frac{186}{70,423}$
16	70,330	485	0.01	0.99	70,087	816,100	11.60	0.01
17	69,844	1,021	0.01	0.99	69,334	746,013	10.68	0.01
18	68,824	1,683	0.02	0.98	67,982	676,679	9.83	0.02
19	67,141	2,415	0.04	0.96	65,933	608,697	9.07	0.04
20	64,726	2,771	0.04	0.96	63,340	542,764	8.39	0.04
21	61,955	3,128	0.05	0.95	60,391	479,423	7.74	0.05
22	58,827	3,525	0.06	0.94	57,064	419,033	7.12	0.06
23	55,302	4,087	0.07	0.93	53,258	361,968	6.55	0.08
24	51,215	4,502	0.09	0.91	48,963	308,710	6.03	0.09
25	46,712	4,791	0.10	0.90	44,317	259,746	5.56	0.11
26	41,921	4,898	0.12	0.88	39,472	215,430	5.14	0.12
27	37,023	4,897	0.13	0.87	34,575	175,958	4.75	0.14
28	32,127	4,854	0.15	0.85	29,700	141,383	4.40	0.16
29	27,273	4,369	0.16	0.84	25,088	111,683	4.10	0.17
30	22,904	4,110	0.18	0.82	20,849	86,595	3.78	0.20
31	18,794	3,637	0.19	0.81	16,976	65,746	3.50	0.21
32	15,157	3,250	0.21	0.79	13,532	48,770	3.22	0.24
33	11,907	2,911	0.24	0.76	10,452	35,238	2.96	0.28
34	8,996	2,323	0.26	0.74	7,835	24,786	2.76	0.30
35	6,673	1,855	0.28	0.72	5,746	16,951	2.54	0.32
36	4,818	1,477	0.31	0.69	4,079	11,206	2.33	0.36
37	3,341	1,111	0.33	0.67	2,785	7,126	2.13	0.40
38	2,230	817	0.37	0.63	1,821	4,341	1.95	0.45
39	1,413	574	0.41	0.59	1,126	2,520	1.78	0.51
40	838	377	0.45	0.55	650	1,394	1.66	0.58
41	461	216	0.47	0.53	353	744	1.61	0.61
42	245	117	0.48	0.52	186	391	1.59	0.63
43	128	60	0.47	0.53	98	205	1.60	0.61
44	68	33	0.49	0.51	51	107	1.58	0.65
45	35	14	0.40	0.60	28	56	1.60	0.50
46	21	8	0.37	0.63	17	28	1.34	0.47
47	13	9	0.67	0.33	9	11	0.83	1.00
48	4	4	1.00	0.00	2	2	0.50	2.00

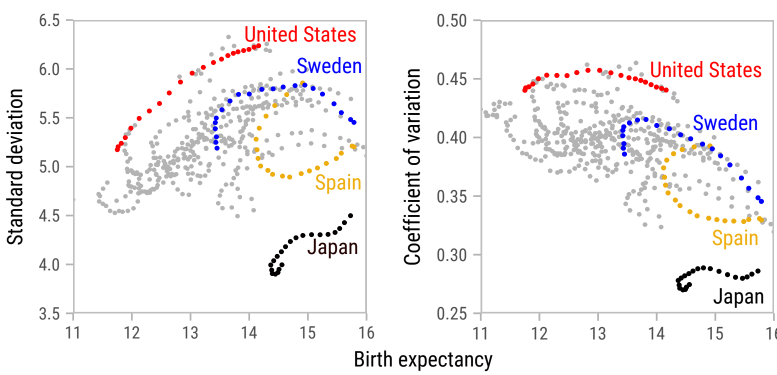
Birth counts retrieved from the Human Fertility Database (2020).

[1] To calculate $_nL_x$ we assume that births occur in the middle of the age interval. Thus, $_nL_x = (n/2)(l_x + l_{x+n})$. This approach is common in calculation of life tables (Preston, Heuveline, and Guillot 2000).

4.2 Illustration of summary measures

Beyond age-specific fertility functions, our birth table provides the necessary information to construct summary measures that describe the distribution of births, such as birth expectancy, standard deviation, and the coefficient of variation. For the example given in Table 1, at the onset age of the reproductive life span ($\alpha = 12$), birth expectancy for the Danish cohort of 1960 is 15.56 years, that is – to-be-born children wait on average about 16 years from the age of first birth in the female population and are born to women of an average age of 28. The standard deviation and the coefficient of variation for the same cohort are 5.45 years and 0.35, respectively. Summary measures like this can be used to study trends over time. For example, Figure 3 illustrates the relationship between life expectancy and measures of variation (standard deviation in the left panel and the coefficient of variation in the right panel) for all the countries available in the Human Fertility Database (2020). This diagram condenses much of the fertility experience of a population and thus is an explicit macro-level tool to detect patterns across populations. Its analysis is useful to show constraints and connections across populations with low versus high birth expectancy, and with fertility concentrated or spread throughout reproductive life. Formal analysis of the relationships depicted in Figure 3 requires application of rigorous statistical tests and scrutiny under meaningful mathematical frameworks, similar to the ones developed for mortality research (Baudisch 2011; Wrycza, Missov, and Baudisch 2015; Aburto et al. 2020; Colchero et al. 2016).

Figure 3: Relationship between birth expectancy and two alternative measures of dispersion in the number of births over the reproductive life span for all the countries included in the Human Fertility Database (2020), Cohorts 1940–1960



5. Discussion

The fact that individuals are born only once is heavily exploited in this paper. We show that shifting the perspective from women to children results in a simple and straightforward framework to calculate the central triplet of survival analysis functions describing age-specific birth patterns. These functions help in comparing birth schedules across populations to probe the underlying mechanisms that drive the dynamics of fertility over time. In general, our approach complements traditional methods in fertility research and can be applied depending on the available data and research question.

We introduce the idea of a birth table that depicts how a to-be-born offspring population is decremented by birth. This concept is different from – but closely linked to – the concept of a life table in fertility research (Hoem 1970; Feeney 1983; Oechsli 1975; Golbeck 1986; Chiang and Van Den Berg 1982). The Human Fertility Database (2020) offers cohort and period fertility tables for a range of countries. Cohort fertility tables model the process of becoming a mother by age and, if data are available, by parity. For each cohort, life table functions, analogous to those found in mortality research (Preston, Heuveline, and Guillot 2000), are computed from fertility rates, which relate birth by age of the mother and child's birth order to the entire female population at a given age (Jasilioniene et al. 2016). Cohort fertility tables, therefore, require information on age-specific birth counts and the female population exposed to the risk of childbearing.

In comparison, our "born once, die once" approach allows computation of birth tables and construction of meaningful summary measures such as birth expectancy and its standard deviation by solely using age-specific birth counts. This provides an advantage over calculating the closely related mean age at birth, which requires additional information about the female population exposed to childbearing.

A limiting factor of our approach is its disconnection between specific women and specific children, which does not allow one to distinguish parity at birth or calculate other measures such as parity-specific fertility rates. Fertility tables based on parity rather than age have previously been developed (Hoem 1970; Feeney 1983; Golbeck 1986). These unconventional fertility tables rely on transition probabilities between stages to calculate average waiting times between birth orders and emphasize the importance of mother's stage rather than age (Caswell et al. 2018). Stage-based fertility tables are related to our approach with respect to calculating waiting times, but differ as they stay within the perspective of an average mother rather than that of a child. Accepting a decrease in sample size, parity-specific results can still be obtained with our approach by limiting the offspring population to children born with a specific parity only.

Another limitation of our approach is that it requires knowledge about completed cohort fertility. Therefore, as it stands, it cannot be used to directly elucidate current fertility trends; nor can it be used to study populations that lack full cohort information. However, similar limitations hold for cohort studies in mortality research. Development of a synthetic cohort concept for fertility would resolve this limitation and thus would be a strong motivation for future research.

The “born once, die once” paradigm offers a novel view on the process of childbearing and allows one to transfer methods from mortality to fertility research. We foresee further research along these lines, where formal demographic methods are applied to our summary measures. For example, it is of great interest to determine the sensitivity of birth expectancy to changes in age-specific birth rates (Keyfitz 1977; Keyfitz and Caswell 2005), deriving analogous relationships to mortality (Wrycza and Baudisch 2012). Likewise, decomposing the change over time in birth expectancy (Vaupel and Canudas-Romo 2003) into different components is a natural extension of our framework. Forecasting techniques applied to mortality hazard and life expectancy might also be useful in forecasting future birth schedules. Reviving the formal relationships from the perspective of a child (Preston 1976; Pearson, Lee, and Bramley-Moore 1899), possibly in connection with the present framework and inspired by novel visualization approaches (Riffe et al. 2019), could be promising. Further extensions of our framework may include a period approach that would define a synthetic cohort of women to a focal offspring population. This would raise a range of novel questions. For example, how to account for childless women or potential quantum-tempo distortions (Bongaarts and Feeney 1998) to properly reflect stable population dynamics and structure in a child-centered framework? We envision our approach as a tool for yet unexplored research directions. It should help answer questions that are yet out of reach or out of thought within the boundaries of current methodology. Whatever the applications, they will have to establish their utility in the future.

6. Conclusion

Birth and death mark the cornerstones of life and, thereby, are naturally connected for the individual. Everyone is born once, and everyone dies once. With this perspective we contribute a common approach to construct formal demographic relationships based on a focal individual that experiences its own unique birth and death. This connection allows investigation of mortality and fertility within the same framework and study of their mutual effects and interactions.

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Appendix

Differences from the approach by Baudisch and Stott (2019)

Though based on the "born once, die once" framework previously advanced by Baudisch and Stott (2019), this paper differs slightly in how it specifies the measures that capture the survivorship concept. As the backbone of the survival, density, and hazard functions, we use age-specific cohort birth counts. In contrast, Baudisch and Stott (2019) use age-specific fertility rates as the main input for their framework, which is tailored to aid comparative studies of birth schedules across a wide range of different species. For species other than *homo sapiens*, data on birth and death are scarce (Conde et al. 2019) and researchers infer mortality rates indirectly using Bayesian methods (Colchero and Clark 2012). For animal and plant populations, actual counts of birth and death are not available in public demographic databases. Comparative studies have to rely on birth and death rates given in the form of life tables (DATLife Database 2020) or as entries in population projection matrices (COMADRE Animal Matrix Database 2020; COMPADRE Plant Matrix Database 2020). Age-specific fertility rates are therefore used as a proxy to study birth schedules across species and as basis for a survivorship concept that is captured by a "birth delay function" in Baudisch and Stott (2019).

In comparison, human demographers can rely on rich data when studying fertility. The Human Fertility Database (2020) provides information on age-specific birth counts by parity, population of females at risk of childbearing, and many other indicators of human fertility. The mathematical formulations developed in this article strive to harvest the detailed data available specifically for the study of human fertility, and hence they differ from the original approach by Baudisch and Stott (2019).

For researchers solely interested in human fertility, we recommend using the approach developed in the present paper. For comparative studies that include non-human birth schedules, we suggest using the formulations advanced in Baudisch and Stott (2019).

Table A-1: Glossary of terms

Notation	Description
X	Random variable denoting the time a child waits to the event of its own birth.
α	Onset age of the reproductive life span.
β	Final age of the reproductive life span.
B	Total size of offspring population.
$b(x)$	Number of babies born to women of age x.
$B(x)$	Cumulative number of babies born to women up to age x.
$S(x)$	*Survivorship function*: Percentage of unborn children to women up to age x.
$h(x)$	*Birth hazard*: Instantaneous rate of being born to women of age x.
$f(x)$	*Probability density function*: Fraction of babies born to women of age x.
$l(x)$	Children left to be born to women age x and above.
${}_nb_x$	Children being born within age range x to $x+n$.
${}_nq_x$	Probability that a child is born within age range x to $x+n$.
${}_np_x$	Probability that a child is not born within age range x to $x+n$.
${}_nL_x$	Person-years spent in the unborn state between age x and $x+n$.
T_x	Remaining person-years above age x.
${}_nm_x$	Birth rate within the focal age group.
$e(\alpha)$	*Birth expectancy*: Mean waiting time to be born.
σ	*Standard deviation*: Spread in the waiting time to be born.