



*Demographic Research* a free, expedited, online journal  
of peer-reviewed research and commentary  
in the population sciences published by the  
Max Planck Institute for Demographic Research  
Konrad-Zuse Str. 1, D-18057 Rostock · GERMANY  
[www.demographic-research.org](http://www.demographic-research.org)

---

**DEMOGRAPHIC RESEARCH**

**VOLUME 14, ARTICLE 7, PAGES 111-138**  
**PUBLISHED 17 FEBRUARY 2006**

<http://www.demographic-research.org/Volumes/Vol14/7/>

DOI: 10.4054/DemRes.2006.14.7

*Research Article*

**The relative tail of longevity and  
the mean remaining lifetime**

**Maxim Finkelstein**

**James W. Vaupel**

© 2006 Max-Planck-Gesellschaft.

## Table of Contents

1	Introduction	112
2	The tail of longevity	114
3	Loaded redundancy	116
4	Unloaded redundancy (repairable systems)	119
5	Heterogeneity	122
6	Mean remaining lifetime function	129
7	Comparing relative tails for redundant objects	130
8	Relative tails in the heterogeneous case	133
9	Concluding remarks	135
10	Acknowledgements	135
	References	136

## **The relative tail of longevity and the mean remaining lifetime**

**Maxim Finkelstein**<sup>1</sup>

**James W. Vaupel**<sup>2</sup>

### **Abstract**

Vaupel (1998) posed the provocative question, “When it comes to death, how do people and flies differ from Toyotas?” He suggested that as the force of natural selection diminishes with age, structural reliability concepts can be profitably used in mortality analysis. Vaupel (2003) went a step further, using simulations to investigate the impact of redundancy, repair capacity, and heterogeneity on the relative length of post-reproductive life spans, called relative tails of longevity. His 2003 paper showed that structural redundancy and the possibility of repair decrease the relative tail of longevity, whereas greater heterogeneity increases it. Here, we consider the problem in much greater generality and prove these results analytically. Structures with repairable and non-repairable components are considered. Heterogeneity is described by a frailty-type model and different definitions of the tail of longevity are discussed.

---

<sup>1</sup> Department of Mathematical Statistics, University of the Free State, PO Box 339, 9300 Bloemfontein, Republic of South Africa, E-mail: FinkelM.SCI@mail.uovs.ac.za, Max Planck Institute for Demographic Research, Konrad-Zuse-Strasse 1,18057 Rostock, Germany.

<sup>2</sup> Max Planck Institute for Demographic Research, Konrad-Zuse-Strasse 1,18057 Rostock, Germany. E-mail: jwv@demogr.mpg.de.

## 1. Introduction

The force of evolution peters out with age. Natural selection determines mortality trajectories of organisms at reproductive ages. But what happens afterwards? What mechanisms are responsible for survival at the post-reproductive ages? We do not observe a ‘wall of death’ at the age at which reproduction ceases; on the contrary, many species experience a remarkably long post-reproductive period with a deceleration, a leveling-off or even a decrease in the respective mortality rates. Vaupel (2003) addressed these questions and drew an analogy from structural reliability concepts. This analogy was used before (see, e.g., Gavrilov and Gavrilova, 2002); however, Vaupel for the first time linked it primarily with the post-reproductive period only. He used the following loose analogy for explaining the connection between longevity and the post-reproductive age: “The speed and trajectory of a ball is governed by the pitcher’s strength and skill up to the moment the ball leaves the pitcher’s hand. Thereafter, the ball’s course is determined by the force of gravity acting on the momentum of the ball. Similarly, the course of life until the end of reproduction is determined by evolutionary forces. After reproduction ceases, the remaining trajectory of life is determined by forces of wear, tear, and repair acting on the momentum produced by the Darwinian forces operating earlier in life”. As the human organism is an extremely complex structure of billions of components, it follows from the structural reliability theory that the only way to survive for a relevant period of time in such a system is to be sufficiently redundant on different levels and to possess the capability of repair.

Thus, the answer to Vaupel’s (1998) provocative question, “When it comes to death, how do people and flies differ from Toyotas?” can be interpreted in the following way: As the force of natural selection diminishes with age, structural reliability concepts can be profitably used in mortality analysis. It means that the design of the structure is more or less fixed at this stage and its evolution in time is governed by reliability laws. However, it does not mean that these concepts cannot be used for mortality modeling at earlier ages, but in this case they should be combined with the laws of natural selection. We hope that the corresponding models can be developed in the nearest future.

Vaupel (2003) used simulations and empirical reasoning to investigate for some specific cases the impact of redundancy, repair capacity, and heterogeneity on the relative length of post-reproductive life spans, called relative tails of longevity. The results were as follows: Redundancy and the possibility of repair decrease the relative tail of longevity, whereas heterogeneity increases it. Some general considerations of reliability theory also support this claim: It is well known (Barlow and Proschan, 1975) that the survival curve of a structure with a higher level of redundancy stays longer at

larger values but then decreases more steeply than a survival curve of a structure with a lower level of redundancy.

The relative tail of longevity is defined via the difference between the quantiles of the corresponding distributions (see Section 2). A comparison between them is not straightforward, as we compare the quantiles of different distributions. Moreover, an increase (decrease) in the distance between the quantiles does not mean automatically an increase (decrease) in the relative distance.

A possible deceleration in mortality rates at old ages, which was already mentioned by Gompertz (1825) and Makeham (1867) and was first modeled via the concept of population heterogeneity by Beard (1959, 1971; see also Vaupel et al. (1979) for a more general frailty model) can help in understanding why heterogeneity increases the tail of longevity. Due to this deceleration, one can expect that the distance between two quantiles of the *same* distribution is higher for populations with a higher level of heterogeneity (with the same baseline mortality rate). But *a priori* it is not clear at all whether this effect is maintained for the relative distance and for different distributions.

Redundancy is a main tool in designing reliable technical structures. The idea that redundant structures constitute a plausible lifetime model seems very attractive, as extremely high ‘reliability of humans’ is likely to exist in nature only with the help of redundancy on different levels. The mortality rates of the simplest redundant structures of identical components with constant mortality rates, operating in parallel, were analyzed by Gavrilov and Gavrilova (1991, 2002). The authors show that for a sufficiently small  $t$ , the mortality rate of the fixed parallel structure (loaded redundancy) approximately follows the power law and the mortality rate of a structure with a random number of initially operable components approximately follows the Gompertz law. The latter can be explained also in terms of a general frailty model (Finkelstein, 2003). The mortality plateau emerges naturally in this specific model, as the mortality rate of a redundant system tends with time to the mortality rate of the last surviving component, which is assumed to be constant. Steinsaltz and Evans (2003) explained mortality plateaus from the more general viewpoint of quasi-stationary distributions (see also Aalen and Gjessing, 2001).

Repair capacity is crucial for the theory of repairable engineering systems. It turns out that this property also plays an important role in modern theories of aging (Kirkwood, 1997; Horiuchi, 2002; Yashin et al., 2000). In the current paper, we consider models of perfect repair, i.e. a component after repair is as good as a new one. In this case, the repair is equivalent to the substitution of the failed component by a spare one and is, in fact, a specific case of redundancy (unloaded).

The goal of our technical paper is to prove the empirical and simulation results of Vaupel (2003) analytically and to generalize them to arbitrary lifetime distributions where possible.

In Section 2, we give formal definitions of the tail of longevity and of the relative tail of longevity. Sections 3 and 4 are devoted to proving that redundancy decreases the relative tail of longevity. As mentioned, unloaded redundancy can be interpreted in terms of repair. Section 5 studies the impact of heterogeneity on the relative tail of longevity. As the comparison of quantiles is parameter-sensitive, we also consider a more traditional measure of the tail of a distribution in Sections 6-8: the mean remaining lifetime function (life expectancy at age  $t$ ). We examine the influence of redundancy and heterogeneity on this function. As expected, redundancy decreases the relative mean remaining lifetime function and heterogeneity increases it. Finally, Section 9 formulates the overall conclusion and also discusses other possible measures of the tail of longevity.

## 2. The tail of longevity

Consider a population of a sufficiently large size  $N$ . Denote by  $X$  a random age at death and by  $\omega_N$  a random maximal age at death (the age at last death) in this population. It is challenging to define a tail of longevity as some remaining potential lifetime, taking into account the maximal lifetime variable  $\omega_N$ . A natural candidate for this is the difference  $D_N = \omega_N - X$ , although it formally allows for negative values.

Denote by  $\tau(\omega_N, q)$  the  $q$ -quantile for the distribution of  $\omega_N$ :  $\Pr(\omega_N \leq \tau(\omega_N, q)) = q$ , and by  $\tau(q_0)$  the  $q_0$ -quantile for the distribution of  $X$ :  $\Pr(X \leq \tau(q_0)) = q_0$ . Vaupel (2003) defines the length of the *tail of longevity* (we will omit the term ‘length’ for brevity sake) as the difference  $\tau(\omega_N, q) - \tau(q_0)$  and the *relative tail of longevity* as  $(\tau(\omega_N, q) - \tau(q_0)) / \tau(q_0)$ . Our main focus in the current paper is on the latter characteristic. Relative measures are necessary for adequate comparisons of tails in different populations.

Vaupel (2003) considered specific values of quantiles:  $q = 0.5$  and  $q_0 = 0.9$ . The latter value marks the left end point of the post-reproductive zone for some organisms, where the force of natural selection is active no longer. The median of the maximal life span distribution  $\tau(\omega_N, 0.5)$  is just a reasonable choice for a quantile of this distribution. Note that formally we do not rely on specific values of  $q$  and  $q_0$ : the only reasonable restriction is that the corresponding quantiles should be properly ordered:  $\tau(\omega_N, q) > \tau(q_0)$ , which is obviously the case in reality.

The cumulative distribution function (Cdf) of age at death  $X$  is defined as:

$$F(t) = 1 - \exp\left(-\int_0^t \mu(u) du\right), \quad (1)$$

where  $\mu(t)$  is the mortality rate.

Let

$$S(t) = N \exp\left(-\int_0^t \mu(u) du\right) \quad (2)$$

be the expected number of members who will survive at  $t$ , starting with initial value  $S(0) = N$ .

Let  $\omega_N$  be the maximal age at death for this sample of size  $N$ . Thatcher (1999) showed that the Cdf of  $\omega_N$  for large  $N$  is:

$$\begin{aligned} F_N(t) &\equiv \Pr(\omega_N \leq t) = (F(t))^N = \left(1 - \frac{S(t)}{N}\right)^N \\ &\approx \exp(-S(t)) = \exp\left(-N \exp\left(-\int_0^t \mu(u) du\right)\right). \end{aligned} \quad (3)$$

Let  $\tau(\omega_N, q)$  denote the  $q$ -quantile:  $\Pr(\omega_N \leq \tau(\omega_N, q)) = q$ . Using eq. (3),  $\tau(\omega_N, q)$  is obtained from:

$$S(\tau(\omega_N, q)) = -\ln q \quad (4)$$

or, using eq. (2):

$$\int_0^{\tau(\omega_N, q)} \mu(u) du \approx \ln N - \ln(-\ln q). \quad (5)$$

The second term on the right in eq. (5) is of minor importance, as  $N$  is large and we are not interested in the ‘too high quantiles’ when studying the maximal value

distributions. For large enough  $N$ , the approximate relation (5) can be practically considered as equality and this will be assumed in what follows.

Doubling the sample size  $N$  will only slightly increase  $\tau(\omega_N, q)$  for sufficiently large  $N$ . The increase from  $N$  to  $N^2$  or  $N^3$  gives a substantial increase, depending on the shape of the mortality rate: It is smaller for increasing failure rates and larger for constant and decreasing failure rates. This result follows from eq. (5). In Table 1 of Vaupel (2003), increasing  $N$  from  $10^3$  to  $10^6$  extends the median maximal lifespan for the constant mortality rate from 73 to 142 years, which agrees with eq. (5). Increasing  $N$  from  $10^6$  to  $10^9$  increases the median maximal lifespan from 142 to 211 years, which also matches eq. (5).

Our goal is to compare  $\tau(\omega_N, q)$  with the quantile  $\tau(q_0)$  obtained from Cdf (1):  $F(\tau(q_0)) = q_0$ . The quantile  $\tau(q_0)$ , chosen as 0.9, defines the starting point of old age (Vaupel, 2003; ten percent of the population alive at age  $\tau(0.9)$ ). However, formally we are not very concerned with the concrete values of  $q_0$  and  $q$ , as we only need the ordering:  $\tau(q_0) < \tau(\omega_N, q)$ .

Vaupel (2003) defines the tail of longevity as:

$$TL(q, q_0) \equiv \tau(\omega_N, q) - \tau(q_0) \quad (6)$$

and the relative tail of longevity as

$$RTL(q, q_0) \equiv \frac{\tau(\omega_N, q)}{\tau(q_0)} - 1. \quad (7)$$

The influence of redundancy and heterogeneity on these characteristics will be studied in the next three sections.

### 3. Loaded redundancy

Consider the loaded redundancy when  $n$  statistically independent, identical components in parallel, operating simultaneously, constitute a system with the Cdf of time at death (failure):

$$F_n(t) = (F(t))^n; n = 1, 2, \dots \quad (8)$$

The corresponding quantile  $\tau(n, q_0)$ ;  $\tau(1, q_0) \equiv \tau(q_0)$  is obtained from:  $F_n(\tau(n, q_0)) = q_0$ , or equivalently:

$$F(\tau(n, q_0)) = q_0^{\frac{1}{n}}. \quad (9)$$

It means that the effect of redundancy of this type changes the baseline level  $q_0$  into  $q_0^{\frac{1}{n}}$ . For reasonable parameter values this usually leads to a substantial increase of the quantile.

What about the maximal lifespan quantile? The only difference from the baseline  $\tau(\omega_N, q)$  is the size of the sample, which is now  $nN$ , because the maximal value is observed at the failure of the last of the  $nN$  components. Therefore, eq. (5) for obtaining  $\tau(\omega_{nN}, q)$  turns into

$$\int_0^{\tau(\omega_{nN}, q)} \mu(u) du = \ln N + \ln n - \ln(-\ln q) \quad (10)$$

for obtaining  $\tau(\omega_{nN}, q)$ . Usually,  $n$  is small with respect to  $N$  (although probably this is not the case for the molecular or genetic level). Eq. (10) is asymptotic as  $N \rightarrow \infty$ , and the quantile  $\tau(\omega_{nN}, q)$  depends on  $N$ , so that the term  $\ln n$  is negligible:

$$\tau(\omega_{nN}, q) / \tau(\omega_N, q) \rightarrow 1 \text{ as } N \rightarrow \infty. \quad (11)$$

**Proposition 1.** Let sample size  $N$  be sufficiently large. Then the relative tail of longevity for a system with a loaded redundancy structure is smaller than the one for a non-redundant system:

$$RTL(n, q, q_0) < RTL(q, q_0); n = 2, 3, \dots \quad (12)$$

*Proof:* it follows from eq. (11) that for  $N$  large enough:

$$\frac{\tau(n, q_0)}{\tau(q_0)} > \frac{\tau(\omega_{nN}, q)}{\tau(\omega_N, q)} \quad (13)$$

and, in accordance with the definition of the relative tail of longevity in eq. (7):

$$\frac{RTL(n, q, q_0) + 1}{RTL(q, q_0) + 1} = \frac{\tau(\omega_{nN}, q)\tau(q_0)}{\tau(n, q_0)\tau(\omega_N, q)} < 1,$$

which leads to inequality (12). ♦

Similarly for  $N$  large enough:

$$\frac{\tau(\omega_{(n+1)N}, q)\tau(n, q_0)}{\tau(n+1, q_0)\tau(\omega_{nN}, q)} < 1; n = 1, 2, \dots, \quad \tau(1, q_0) \equiv \tau(q_0), \quad (14)$$

which means that the relative tail of longevity decreases with  $n$ .

Inequalities (13) and (14) hold for reasonable values of the parameters and show that the loaded redundancy decreases the relative tail of longevity.

**Example 1.** Consider the exponential case  $\mu(t) = \mu$  and the level of redundancy  $n = 2$ . From eq. (10):

$$\tau(\omega_{2N}, q) = \frac{\ln N + \ln 2 - \ln(-\ln q)}{\mu}.$$

Using eq. (5):

$$\frac{\tau(\omega_{2N}, q)}{\tau(\omega_N, q)} = \frac{\ln N + \ln 2 - \ln(-\ln q)}{\ln N - \ln(-\ln q)} \approx 1 + \frac{\ln 2}{\ln N}. \quad (15)$$

From eq. (9):

$$\tau(2, q_0) = \frac{-\ln(1 - \sqrt{q_0})}{\mu}.$$

Therefore:

$$\frac{\tau(2, q_0)}{\tau(q_0)} = \frac{\ln(1 - \sqrt{q_0})}{\ln(1 - q_0)}. \quad (16)$$

Thus, for  $N = 10^6$ ,  $q_0 = 0.9, q = 0.5$  eq. (15) gives 1.05 while equation (16) gives 1.3. For the relative tails:

$$RTL(q, q_0) \equiv \frac{\tau(\omega_N, q)}{\tau(q_0)} - 1 = \frac{\ln N - \ln(-\ln q)}{-\ln(1 - q_0)} - 1 \approx 5.15,$$

$$RTL(2, q, q_0) \equiv \frac{\tau(\omega_{2N}, q)}{\tau(2, q_0)} - 1 = \frac{\ln N + \ln 2 - \ln(-\ln q)}{-\ln(1 - \sqrt{q_0})} - 1 \approx 3.95,$$

which shows a decrease of approximately 30% of the relative tail of longevity.

#### 4. Unloaded redundancy (repairable systems)

Consider the unloaded redundancy when one of the identical components starts operating and the other  $n-1$  are in stand by. As the operating one fails, it is immediately replaced by the stand by one etc. The system fails when the last component fails. This is interpreted as the perfect repair of the failed object and we shall use this interpretation in what follows.

For a constant mortality rate of a component:  $F(t) = 1 - \exp(-\mu t)$ , the probability of the system failure at  $t$  is an  $n-1$  fold convolution of the exponential distribution:

$$F_n(t) = 1 - \exp(-\mu t) \sum_0^{n-1} \frac{(\mu t)^k}{k!}; n = 1, 2, \dots$$

The mortality rate of the system is (Barlow and Proschan, 1975):

$$\mu_n(t) = \frac{\mu^n t^{n-1}}{(n-1)! \sum_0^{n-1} \frac{(\mu t)^k}{k!}}. \quad (17)$$

$\mu_n(t), n > 1$  is a monotonically increasing function,  $\mu_n(0) = 0$ ;  $\mu_n(t) \rightarrow \mu$  as  $t \rightarrow \infty$  and

$$\mu_n(t) < \mu_{n-1}(t), \forall t > 0; n > 1. \quad (18)$$

(Barlow and Proschan, 1975). The  $q_0$ -quantiles for the simple and redundant cases are defined by equations:

$$\mu \tau(q_0) = -\ln(1 - q_0), \quad (19)$$

$$\int_0^{\tau(n, q_0)} \mu_n(u) du = -\ln(1 - q_0), \quad (20)$$

respectively. Due to inequality (18):

$$\tau(n, q_0) > \tau(n-1, q_0); n > 1, \quad (21)$$

which implies:

$$t(n, q_0) > t(1, q_0) \equiv t(q_0); n > 1. \quad (22)$$

The corresponding  $q$ -quantiles, in accordance with eq. (5), are defined by:

$$\mu \tau(\omega_N, q) = \ln N - \ln(-\ln q), \quad (23)$$

$$\int_0^{\tau(n, \omega_N, q)} \mu_n(u) du = \ln N - \ln(-\ln q), \quad (24)$$

where  $\tau(n, \omega_N, q), n > 1$  denotes the  $q$ -quantile for a redundant object with  $\tau(1, \omega_N, q) \equiv \tau(\omega_N, q)$ . As previously, the sample size  $N$  is supposed to be sufficiently large and the quantiles should be ordered as:

$$-\ln(1 - q_0) < \ln N - \ln(-\ln q). \quad (25)$$

**Proposition 2.** Let the mortality rate of a non-redundant component be constant. Then the relative tail of longevity (for a sufficiently large sample size  $N$ ) is smaller for a system with unloaded redundancy structure than for a non-redundant system.

*Proof.* Under condition (11), we show that inequality (13) holds for the case under consideration, which, using the notation of eqs. (19) to (24), is equivalent to:

$$\frac{\tau(n, q_0)}{-\ln(1 - q_0)} > \frac{\tau(n, \omega_N, q)}{\ln N - \ln(-\ln q)}. \tag{26}$$

Due to the monotonicity of  $\mu_n(t)$  and because for large  $t$  this function approaches the ‘non-redundant’ value  $\mu$ , inequality (26) is achieved for  $N$  large enough. The mortality rate  $\mu(\tau(n, \omega_N, q))$  is sufficiently close to  $\mu$  in this case and, taking into account eqs. (23) and (24),  $\tau(n, \omega_N, q)/(\ln N - \ln(-\ln q))$  is close to 1. Therefore, Proposition 2 holds and the unloaded redundancy also decreases the relative tail of longevity. ♦

Proposition 1 is also explained from the general fact that:

$$\mu_n(t) \rightarrow \mu(t), \quad t \rightarrow \infty, \tag{27}$$

as the mortality rate  $\mu_n(t)$  of a system with a loaded redundancy tends to the mortality rate of the last remaining component. This means that for  $n > 1$ , as  $t \rightarrow \infty$  and  $N \rightarrow \infty$ :

$$\frac{\int_0^{\tau(n, \omega_N, q)} \mu_n(u) du}{\int_0^{\tau(\omega_N, q)} \mu(u) du} = 1 + o(1), \tag{28}$$

which leads to Proposition 1.

Proposition 2 was proved only for a constant mortality rate. Eq. (27) is true for this case. However, eq. (27) does not hold for an arbitrary mortality rate  $\mu(t)$ . The corresponding (random) mortality rate is defined (Aven and Jensen, 1999) as

$$\begin{cases} 0 & t \leq t_{n-1} \\ \mu(t - t_{n-1}), & t > t_{n-1} \end{cases},$$

where  $t_{n-1}$  is the random failure time of the last but one component in the unloaded redundancy structure of  $n$  components. This effect introduces ‘unexpected’ heterogeneity in this case through random  $t_{n-1}$ . The increasing sample size  $N$  decreases the relative tail of longevity, whereas heterogeneity increases it. This needs a more detailed investigation in the future.

## 5. Heterogeneity

Consider the values of the accumulated mortality rate  $M(t)$  on the  $y$ -axis:

$$y = M(t) = \int_0^t \mu(u) du . \quad (29)$$

Denote by  $y_1$  and  $y_2$  specific values of the accumulated mortality rate in the quantile points:  $M(\tau(q_0))$  and  $M(\tau(\omega_N, q))$ , respectively. It follows from eq. (1) and eq. (5) that

$$y_1 = M(\tau(q_0)) = -\ln(1 - q_0), \quad y_2 = M(\tau(\omega_N, q)) = \ln N - \ln(-\ln q) . \quad (30)$$

We shall prove the following hypothesis of Vaupel (2003):

*-The more variability (environmental or internal) in a mortality pattern of an object, the longer its relative tail of longevity.*

We start with a simple case. Consider an object in a baseline environment with a constant mortality rate  $\mu$ . Assume that some perturbation of a baseline environment obeys a multiplicative frailty model:

$$\mu(t, Z) = Z\mu , \quad (31)$$

where  $Z$  is a random variable with Cdf  $L(z)$ , support  $[a, b]; 0 \leq a < b \leq \infty$ , and  $E(Z) = 1$ , which means that  $a < 1 < b$ . We shall compare tails for a constant mortality rate  $\mu$  and a stochastic mortality rate (31). The mixture (observed) mortality rate  $\mu_m(t)$ , which corresponds to eq. (31), monotonically decreases to the mortality rate of the strongest population  $a\mu$  with an initial level defined as  $\mu_m(0) = \mu$ . (Finkelstein and Esaulova, 2001).

Denote by  $\tau(\mu, \omega_N, q)$ ,  $\tau(\mu_m, \omega_N, q)$  the  $q$ -quantiles for samples of size  $N$  for mortality rates  $\mu$  and  $\mu_m(t)$ , respectively, and the corresponding 'ordinary'  $q_0$ -quantiles-by  $\tau(\mu, q_0)$  and  $\tau(\mu_m, q_0)$ . In accordance with eq. 30, define two quantiles for each curve  $\mu$  and  $\mu_m(t)$ :

$$\mu \tau(\mu, q_0) = y_1, \quad \int_0^{t(\mu_m, q_0)} \mu_m(u) du = y_1, \quad (32)$$

$$\mu \tau(\mu, \omega_N, q) = y_2, \quad \int_0^{\tau(\mu_m, \omega_N, q)} \mu_m(u) du = y_2.$$

Define the tail of longevity for  $\mu$  constant as a baseline one. When changing from  $\mu$  to  $\mu_m(t)$ , consider the increments at the right and left ends of this tail, respectively:

$$\begin{aligned} \Delta \tau_{right}(\mu_m, \mu) &\equiv \tau(\mu_m, \omega_N, q) - \tau(\mu, \omega_N, q) > 0, \\ \Delta \tau_{left}(\mu_m, \mu) &\equiv \tau(\mu_m, q_0) - \tau(\mu, q_0) > 0. \end{aligned}$$

Due to eqs. (25), (30), and (32) and, taking into account that  $\mu_m(t)$  monotonically decreases:

$$\Delta \tau_{right}(\mu_m, \mu) - \Delta \tau_{left}(\mu_m, \mu) > 0, \quad (33)$$

which means that heterogeneity increases *the tail of longevity*, defined by relation (6):

$$\begin{aligned} TL(\mu_m, q, q_0) &\equiv \tau(\mu_m, \omega_N, q) - \tau(\mu_m, q_0) \\ &= \tau(\mu, \omega_N, q) + \Delta \tau_{right}(\mu_m, \mu) - \tau(\mu, q_0) - \Delta \tau_{left}(\mu_m, \mu) \\ &\equiv TL(\mu, q, q_0) + \Delta \tau_{right}(\mu_m, \mu) - \Delta \tau_{left}(\mu_m, \mu), \end{aligned}$$

so that:

$$TL(\mu_m, q, q_0) > TL(\mu, q, q_0). \quad (34)$$

This is not, in fact, surprising: As the weakest populations are dying out first (Vaupel *et al.*, 1979), the ‘homogeneous’ mortality rate  $\mu$  is ‘bent down’ and the survival probability is higher.

Inequality (33) does not guarantee a similar ordering of *relative tails of longevity*, as the following inequality should hold for this property:

$$\frac{(\tau(\mu, \omega_N, q) + \Delta\tau_{right}(\mu_m, \mu))\tau(\mu, q_0)}{\tau(\mu, \omega_N, q)(\tau(\mu, q_0) + \Delta\tau_{left}(\mu_m, \mu))} > 1, \quad (35)$$

which can be seen taking into account definitions of  $\Delta\tau_{left}(\mu_m, \mu)$  and  $\Delta\tau_{right}(\mu_m, \mu)$ . Inequality (35) is equivalent to:

$$\frac{\Delta\tau_{right}(\mu_m, \mu)}{\tau(\mu, \omega_N, q)} > \frac{\Delta\tau_{left}(\mu_m, \mu)}{\tau(\mu, q_0)}. \quad (36)$$

Although inequality (33) holds, the quantiles are ordered as:  $\tau(\mu, \omega_N, q) > \tau(\mu, q_0)$ , which can change the sign of inequality for relative tails of longevity. Consider a simple illustrative example, which shows that relations (34) and  $\mu_m(t) < \mu(t); t > 0$  do not guarantee an increase in the relative tail of longevity.

**Example 2.** Let  $\mu(t) = \mu_1; \mu_m(t) = \mu_2$  and  $\mu_2 < \mu_1$ . It does not matter that this situation does not model the frailty setting. It is important that  $\mu_m(t) < \mu(t), t > 0$  and that the tail of longevity is larger for the Cdf defined by  $\mu_2$  than for the Cdf defined by  $\mu_1$ . For this case:

$$\tau(\mu_1, q_0) = \frac{y_1}{\mu_1}; \quad \tau(\mu_1, \omega_N, q) = \frac{y_2}{\mu_1}, \quad (37)$$

$$\tau(\mu_2, q_0) = \frac{y_1}{\mu_2}; \quad \tau(\mu_2, \omega_N, q) = \frac{y_2}{\mu_2} \quad (38)$$

and

$$\begin{aligned} \Delta\tau_{right}(\mu_2, \mu_1) &= y_2 \left( \frac{1}{\mu_2} - \frac{1}{\mu_1} \right), \\ &> \Delta\tau_{left}(\mu_2, \mu_1) = y_1 \left( \frac{1}{\mu_2} - \frac{1}{\mu_1} \right). \end{aligned}$$

Finally:

$$\frac{\Delta\tau_{right}(\mu_2, \mu_1)}{\tau(\mu_1, \omega_N, q)} = \left( \frac{1}{\mu_2} - \frac{1}{\mu_1} \right) \mu_1 = \frac{\Delta\tau_{left}(\mu_2, \mu_1)}{\tau(\mu_1, q_0)}.$$

We have obtained an equality and not an inequality as in eq. (36).

**Theorem 1.** Under the assumptions defining the frailty model (31), the relative tail of longevity, defined for the Cdf with mortality rate  $\mu_m(t)$ , is larger than the baseline relative tail of longevity, defined for a constant  $\mu$ .

Proof. It follows from eq. (32):

$$\frac{\tau(\mu, \omega_N, q)}{\tau(\mu, q_0)} = \frac{y_2}{y_1}. \quad (40)$$

The corresponding quantiles for a given  $y$  can be obtained from line  $y = \mu t$ , and the relative tail of longevity does not depend on  $\mu$  in this case. Consider now  $\tau(\mu_m, \omega_N, q) / \tau(\mu_m, q_0)$ . The mixture mortality rate  $\mu_m(t)$  in the model defined by eq. (31) decreases monotonically to the mortality rate of the strongest population  $a\mu$ , and the initial level is defined as  $\mu_m(0) = \mu$ . Accumulated mortality  $M_m(t)$  in this case is a convex function, as  $M_m''(t) = \mu_m'(t) < 0; t \geq 0$ . Define rate  $\mu(y_1)$  as

$$\mu(y_1) = \frac{y_1}{t(\mu_m, y_1)} \quad (41)$$

and consider the hypothetical Cdf with this rate  $a < \mu(y_1) < \mu$ . In accordance with eq. (40), the corresponding relative tail of longevity is equal to  $(y_2 / y_1) - 1$  and does not depend on the value of  $\mu$ . As  $M_m(t)$  is convex:

$$\tau(\mu_m, \omega_N, q) > \tau(\mu(y_1), \omega_N, q), \quad (42)$$

which completes the proof. ♦

Similarly, an increase in heterogeneity (e.g., in variance) increases the relative tail of longevity.

Consider now the time dependent baseline rate in eq. (31):

$$\mu(t, Z) = Z\mu(t), \quad (43)$$

For the proper ordering of relative tails we impose natural conditions on  $M(t) = \int_0^t \mu(u)du$  and  $M_m(t) = \int_0^t \mu_m(u)du$  as functions of  $t$ . Inequality (35), written in a standard form:

$$\frac{\tau(\mu(t), \omega_N, q)}{\tau(\mu(t), q_0)} < \frac{\tau(\mu_m(t), \omega_N, q)}{\tau(\mu_m(t), q_0)} \quad (44)$$

is rearranged as:

$$\frac{\tau(\mu_m(t), q_0)}{\tau(\mu(t), q_0)} < \frac{\tau(\mu_m(t), \omega_N, q)}{\tau(\mu(t), \omega_N, q)}. \quad (45)$$

Eq. (45) means that the relative horizontal distance (or relative  $t$ -distance) between the curves  $M_m(t)$  and  $M(t)$  is larger for larger values of  $y = M_m(t) = M(t)$  (for fixed  $y_1, y_2$  and  $y_2 > y_1$ ).

Define by  $M^{-1}(y)(M_m^{-1}(y))$  the corresponding inverse function:  $M(M^{-1}(y)) = y$ .

**Proposition 3.** If the relative  $t$ -distance between the curves  $M_m(t)$  and  $M(t)$ :

$$M(\mu(t), \mu_m(t), y) = \frac{M_m^{-1}(y)}{M^{-1}(y)}, \quad (46)$$

increases in  $y > 0$ , then inequality (45) and the relative tails of longevity ordering of eq. (35) hold.

**Remark.** The relative  $t$ -distance is equal to  $M(\mu(t), \mu_m(t), y) - 1; y \geq 0$ . For simplicity of notation we call  $M(\mu(t), \mu_m(t), y)$  the relative  $t$ -distance (without subtracting 1).

The next theorem shows that the frailty setting of eq. (43) for the specific case when  $\mu(t)$  is a power function, results in an increased relative tail of longevity. As in

Theorem 1, we will not use the specific form of the mixing distribution. But first, we need to formulate two important supplementary results:

**Proposition 4.** Let  $E(Z|t)$  denote the conditional expectation of  $Z$  (on condition that an object did not die in  $[0,t]$ ;  $E(Z|0) \equiv E(Z) = 1$ ). Applying the operation of conditional expectation to both sides of eq. (43):

$$\mu_m(t) = \mu(t)E(Z|t). \quad (47)$$

Then, (Finkelstein and Esaulova, 2001),  $E(Z|t)$  decreases with age  $t$ . (See Yashin and Manton, 1997 for a more general case.)

**Proposition 5.** The relative  $y$ -distance  $M(t)/M_m(t)$ , defined for the setting of Proposition 4, increases.

*Proof:*

$$\left(\frac{M(t)}{M_m(t)}\right)' = \frac{\mu(t)\left(\int_0^t \mu(u)E(Z|u)du - E(Z|t)\int_0^t \mu(u)du\right)}{\left(\int_0^t \mu(u)E(Z|u)du\right)^2} > 0, \quad (48)$$

where the fact that  $E(Z|t)$  decreases is used.

**Theorem 2.** If heterogeneity is described by the multiplicative frailty model of eq. (43), with  $\mu(t) = (\alpha + 1)t^\alpha$ ,  $\alpha > 0$  (Weibull Cdf) and  $Z \in [a,b]$ ;  $0 \leq a < b \leq \infty$ ,  $E[Z] = 1$ , then the relative  $t$ -distance  $M(\mu(t), \mu_m(t), y)$  increases in  $y$ .

*Proof.* Consider a baseline curve  $M(t) = t^{\alpha+1}$ , which defines the family of curves  $\{M(t)_c\} = cM(t)$ ;  $a \leq c \leq 1$ . The relative  $y$ -distance between  $M(t)$  and  $cM(t)$  is constant and equal to  $c$ . The corresponding inverse functions are defined by:

$$M^{-1}(y) = y^{\frac{1}{\alpha+1}} \quad \text{and} \quad M_c^{-1}(y) = c^{-\frac{1}{\alpha+1}} y^{\frac{1}{\alpha+1}},$$

respectively, where the family of inverse curves is denoted by  $\{M_c^{-1}(y)\}$ . Due to this specific form, the relative  $t$ -distance, defined for this case as  $M_{c_1}^{-1}(y)/M_{c_2}^{-1}(y); a \leq c_1, c_2 \leq 1$ , for each pair of curves from  $\{M_c^{-1}(y)\}$  is constant in  $y$ . Therefore, the relative  $t$ -distance between  $M(t)$  and  $cM(t)$  is equal to  $c^{-\frac{1}{\alpha+1}}$  and increases as  $c$  is decreased. From Proposition 5,  $M_m(t)/M(t)$  decreases in  $t$ . As  $t$  increases,  $M_m(t)$  crosses the curves with smaller  $c$ . A similar effect takes place with inverse functions: As  $y$  increases, function  $M_m^{-1}(y)$  ‘climbs’ on a larger  $c^{-\frac{1}{\alpha+1}}$  and  $M(\mu(t), \mu_m(t), y)$  increases. ♦

Proposition 5 states that the relative  $y$ -distance is an increasing function. The similar procedure for the  $t$ -distance results in the following condition:

$$\begin{aligned} & \left( M_m^{-1}(y) / M^{-1}(y) \right)' > 0 \\ \Rightarrow & \frac{M^{-1}(y)}{E(Z | M_m^{-1}(y)) \mu(M_m^{-1}(y))} - \frac{M_m^{-1}(y)}{\mu(M^{-1}(y))} > 0, \end{aligned} \quad (49)$$

Condition (49), as follows from Proposition 3, guarantees the relative tails of longevity ordering of eq. (35). It can be verified for various specific cases such as the power law for the mortality rate and the gamma Cdf for the mixing distribution. As another illustration,  $M(\mu(t), \mu_m(t), y)$  increases for the exponential baseline mortality rate and the gamma frailty model (Vaupel, 2003; Finkelstein and Esaulova, 2001):

**Example 3.** For  $Z$  exponentially distributed with parameter  $\vartheta=1$  and

$$\mu(t, Z) = Zk \exp(t),$$

where  $k > 0$  is a constant (Finkelstein and Esaulova, 2001):

$$\mu_m(t) = \frac{k \exp(t)}{k \exp(t) - k + 1} = 1 + \frac{k - 1}{k \exp(t) - k + 1}$$

and  $M(\mu(t), \mu_m(t), y)$  increases. For  $k = 1$  the observed rate  $\mu_m(t)$  is equal to 1.

## 6. Mean remaining lifetime function

Another tail characteristic can be considered as a reasonable alternative to the tail of longevity: life expectancy at age  $t$ . In accordance with reliability terminology, this function is also called the remaining lifetime (MRL) function. Demote it by  $m(t)$ . It is well known that:

$$m(t) \equiv \frac{\int_t^{\infty} \bar{F}(u) du}{\bar{F}(t)} = \int_0^{\infty} \exp\left(-\int_t^{x+t} \mu(u) du\right) dx. \quad (50)$$

The function  $m(t)$  is a conditional tail measure (on condition that an object did not fail in  $[0, t)$ ). The MRL function uniquely defines the corresponding Cdf  $F(t)$  (Finkelstein, 2002).

Differentiating the right hand side in eq (50):

$$m'(t) = \frac{\mu(t) \int_t^{\infty} \bar{F}(u) du - \bar{F}(t)}{\bar{F}(t)} = \mu(t)m(t) - 1$$

or:

$$\mu(t) = \frac{m'(t) + 1}{m(t)}. \quad (51)$$

Eq. (51) is helpful for comparing shapes of  $\mu(t)$  and  $1/m(t)$  (Mi, 1995; Gupta and Akman, 1995; Finkelstein, 2002).

Let  $F(t), G(t)$  be two lifetime distributions with mortality rates  $\mu_f(t), \mu_g(t)$ , and the MRL functions  $m_f(t), m_g(t)$ , respectively. Define the MRL distance  $D_{mrl}(t)$  as

$$D_{mrl}(t) = |m_f(t) - m_g(t)|; \forall t \in [0, \infty). \quad (52)$$

Comparing relative values is usually more appropriate than comparing absolute values. Define the relative MRL function  $m_r(t)$ , called the *relative tail*, as

$$m_r(t) = \frac{m(t)}{m(0)}. \quad (53)$$

For aging distributions (with decreasing  $m(t)$ ) the function  $m_r(t)$  represents the proportion of the remaining lifetime at age  $t$ . We say that lifetime  $X$  (with the Cdf  $F(t)$ ) is smaller than lifetime  $Y$  (with the Cdf  $G(t)$ ) in the relative MRL ordering and write  $X \leq_{rml} Y$ , if

$$m_{rf}(t) \leq m_{rg}(t); \forall t \in [0, \infty). \quad (54)$$

We now compare the relative tails for redundant structures (see Shaked and Shantikumar, 1993).

## 7. Comparing relative tails for redundant objects

Consider the following quotient:

$$R(t) \equiv \frac{m_{rf}(t)}{m_{rg}(t)} = \frac{m_g(0)}{m_f(0)} \times \frac{m_f(t)}{m_g(t)}. \quad (55)$$

$R(0) = 1$ , and the shape of  $R(t)$  for  $t > 0$  depends on the shapes of  $m_f(t)$  and  $m_g(t)$ .

**Example 4. Unloaded redundancy.** We compare the relative tails for two objects with the constant mortality rates of components and different levels of redundancy:  $m < n$ . Mortality rates  $m_f(t)$  and  $m_g(t)$  for this case are defined by eq. (17):

$$\frac{\mu^i t^{i-1}}{(i-1)! \sum_0^{i-1} \frac{(\mu t)^k}{k!}},$$

for  $i = n$  and  $m$ , respectively.

As the rate of the Erlangian distribution increases monotonically to approach  $\mu$  from below as  $t \rightarrow \infty$ , the functions  $m_f(t)$  and  $m_g(t)$  decrease and  $m_f(t) > m_g(t); t > 0$ . Taking into account that  $m_f(0) = n/\mu$  and  $m_g(0) = m/\mu$ :

$$m_{f'}(t) = \frac{\mu m_f(t)}{n}; \quad m_{r_g}(t) = \frac{\mu m_g(t)}{m}; \quad m < n \quad (57)$$

and

$$R(t) = \frac{m_f(t)m}{nm_g(t)} \rightarrow \frac{m}{n} \quad \text{as } t \rightarrow \infty \quad (58)$$

because  $m_f(t) \rightarrow 1/\mu$  and  $m_g(t) \rightarrow 1/\mu$  as  $t \rightarrow \infty$ . Conditional probability and eq. (50) imply:

$$m_f(t) = \frac{\frac{\exp(-\mu t)}{\mu} \sum_0^{n-1} (n-i) \frac{(\mu t)^i}{i!}}{\exp\{-\mu t\} \sum_0^{n-1} \frac{(\mu t)^i}{i!}}; \quad m_g(t) = \frac{\frac{\exp(-\mu t)}{\mu} \sum_0^{m-1} (m-i) \frac{(\mu t)^i}{i!}}{\exp\{-\mu t\} \sum_0^{m-1} \frac{(\mu t)^i}{i!}} \quad (59)$$

and

$$R(t) = \frac{\sum_0^{n-1} (n-i) \frac{(\mu t)^i}{i!}}{\sum_0^{m-1} (m-i) \frac{(\mu t)^i}{i!}} \times \frac{\sum_0^{m-1} \frac{(\mu t)^i}{i!}}{\sum_0^{n-1} \frac{(\mu t)^i}{i!}} \times \frac{m}{n} \equiv R_1(t) \times R_2(t) \times \frac{m}{n}. \quad (60)$$

$R_1(t)$  on the right in eq. (60) increases in  $t$ ,  $R_2(t)$  decreases and  $R_1(0) = n/m$ ;  $R_2(0) = 1$ ,  $R_1(t) \rightarrow t^{n-m}$ ;  $R_2(t) \rightarrow t^{m-n}$ , as  $t \rightarrow \infty$ . The derivative  $R_2'(t)$  is negative when

$$\sum_{i=0}^{n-1} \frac{(\mu t)^{i+(m-1)}}{i!(m-1)!} - \sum_{j=0}^{m-1} \frac{(\mu t)^{j+(n-1)}}{j!(n-1)!} \geq 0 \quad (61)$$

Changing the index of summation in the second sum:  $j = i + m - n$ , on the left in eq. (61) turns into:

$$\sum_{i=0}^{n-1} \frac{(\mu t)^{i+(m-1)}}{i!(m-1)!} - \sum_{i=n-m}^{n-1} \frac{(\mu t)^{i+(m-1)}}{(i+m-n)!(n-1)!},$$

which is positive, as  $n > m$  and  $i \leq n-1$ . Function  $R_1(t)$  is analyzed similarly. The initial behavior of  $R(t)$  is defined by  $R'_1(0) = \mu(n-m) > 0$ ,  $R'_2(0) = 0$ . This function increases from level  $R(0) = 1$ , then decreases, crossing line  $y=1$  at some  $t_m$ , and approaches  $m/n$  as  $t \rightarrow \infty$  from above.

Therefore, the relative MRL ordering (54) holds for  $t \geq t_m$  so that the *increase in redundancy leads to a decrease in relative tails*.

When  $m = 1$ :

$$R(t) = \frac{m_f(t)\mu}{n} \rightarrow \frac{1}{n}$$

and this function decreases for all  $t \geq 0$ .

**Example 5. Loaded redundancy.** In this case:

$$F(t) = (1 - \exp(-\mu t))^n \quad G(t) = (1 - \exp(-\mu t))^m; \quad m < n.$$

The shape of  $R(t)$  is similar to the one for unloaded redundancy and

$$R(t) \rightarrow \frac{1/m + 1/(m-1) + \dots + 1}{1/n + 1/(n-1) + \dots + 1} \text{ as } t \rightarrow \infty.$$

For  $m = 1$ , the function  $R(t)$  decreases in  $[0, \infty)$ .

## 8. Relative tails in the heterogeneous case

Let  $F(t)$  denote the Cdf of a life span in some deterministic baseline environment and  $G(t)$  - the Cdf in the heterogeneous case. Consider the frailty model of eq. (43) ( $E(Z) = 1, a < 1 < b$ ) and denote:

$$\mu_f(t) \equiv \mu_m(t); \mu_g(t) \equiv \mu(t).$$

Let  $\pi(z|t), t \geq 0$  be the conditional mixing probability density function (pdf):  $\pi(z|0) \equiv \pi(z)$ , where  $\pi(z)$  is the pdf of a random variable  $Z$ . This model describes the influence of heterogeneity on the observed rate  $\mu_m(t)$ . Using the pdf  $\pi(z|t)$ , eq. (47) reads:

$$\mu_m(t) = \mu(t) \int_a^b z \pi(z|t) dz = \mu(t) E(Z|t). \quad (62)$$

Denote by  $m(t, z)$  the MRL function defined by the mortality rate  $\mu(t, z)$  (eq. (43)). The ‘observed MRL function’  $m_m(t)$  is related to  $m(t, z)$  through:

$$m_m(t) = \int_a^b m(t, z) \pi(z|t) dz. \quad (63)$$

**Example 6.** Let  $F(t)$  be an exponential Cdf with parameter  $\mu$ . The observed mortality rate  $\mu_m(t)$  in this case decreases, monotonically converging to the failure rate of the strongest population:

$$\lim_{t \rightarrow \infty} \mu_m(t) = a\mu.$$

Therefore:

$$\lim_{t \rightarrow \infty} m_m(t) = \frac{1}{a\mu}.$$

As  $m(t) = m(0) = 1/\mu$  and

$$m_m(0) = \int_a^b \frac{1}{z\mu} \pi(z) dz = \frac{1}{\mu} E\left(\frac{1}{Z}\right),$$

as  $t \rightarrow \infty$ , we obtain:

$$\frac{m_m(t)}{m(t)} \rightarrow \frac{1}{a} > 1;$$

$$m_{rm}(t) \equiv \frac{m_m(t)}{m_m(0)} \rightarrow \frac{1}{aE\left(\frac{1}{Z}\right)} > 1.$$

The relative tail in the heterogeneous case is equal to 1 at  $t=0$  and increases monotonically to value  $1/(aE(1/Z))$ , whereas the relative tail for the baseline Cdf is constant and equal to 1.

We generalize this result to arbitrary increasing mortality rates:

**Theorem 3.** *In a heterogeneous case modeled by the frailty model of eq. (43), where  $E(Z) = 1$ , the function:*

$$R(t) = \frac{m(0)}{m_m(0)} \times \frac{m_m(t)}{m(t)}$$

increases with  $t$ .

This means that heterogeneity increases the relative tail.

*Proof.* Using definitions (50), (55), and eq. (62):

$$\begin{aligned} R(t) &= \frac{\int_0^{\infty} \exp\left(-\int_t^{x+t} \mu_m(u) du\right) dx \int_0^{\infty} \exp\left(-\int_0^x \mu(u) du\right) dx}{\int_0^{\infty} \exp\left(-\int_0^x \mu_m(u) du\right) dx \int_0^{\infty} \exp\left(-\int_t^{x+t} \mu(u) du\right) dx} \\ &= \frac{\int_0^{\infty} \exp\left(-\int_t^{x+t} E[Z | u] \mu(u) du\right) dx \int_0^{\infty} \exp\left(-\int_0^x \mu(u) du\right) dx}{\int_0^{\infty} \exp\left(-\int_0^x E[Z | u] \mu(u) du\right) dx \int_0^{\infty} \exp\left(-\int_t^{x+t} \mu(u) du\right) dx} \end{aligned} \tag{64}$$

It is sufficient to show that

$$B(t, x) \equiv \frac{\int_0^{\infty} \exp\left(-\int_t^{x+t} E[Z|u] \mu(u) du\right) dx}{\int_0^{\infty} \exp\left(-\int_t^{x+t} \mu(u) du\right) dx}$$

increases in  $t$ . As  $E(Z|t)$  decreases in age (Proposition 4),  $B'_t(t, x) > 0$  and  $B(t, x)$  increases with  $t$ . ♦

## 9. Concluding remarks

Our study shows that structural reliability concepts suggested in Vaupel (1998) can be really helpful for analyzing trajectories of mortality at the post-reproductive period. We prove analytically that the properties of the relative tail of longevity, described in Vaupel (2003) using simulation results, are valid in a much greater generality. Namely: Structural redundancy and the possibility of repair decrease the relative tail of longevity, whereas greater heterogeneity increases it. These properties are important for analyzing the nature of mortality curves at advanced ages.

As mentioned in the Introduction, other measures driven by the difference  $D_N = \omega_N - X$  can be considered also. For instance, another possible natural relative measure is  $(E[\omega_N] / \tau(q_0)) - 1$ . It is clear, however, that this is just a specific case of our measure (7), as the value of  $q$  can be chosen as a solution of the equation  $E[\omega_N] = \tau(\omega_N, q)$ .

A stochastic analysis of the influence of redundancy, repair capacity, and heterogeneity on the random variable  $(\omega_N / X) - 1$  can be hopefully also performed within the framework of stochastic ordering (Shaked and Shantikumar, 1993), but this is a topic for a future study.

## 10. Acknowledgments

The authors thank two anonymous referees for valuable comments, which greatly improved the presentation of this paper. We also thank S. Baker (Max Planck Institute for Demographic Research) for editing the manuscript.

## References

- Aven, T., and Jensen, U. (1999). *Stochastic Models in Reliability*. Springer.
- Aalen, O.O., and Gjessing, H.K. (2001). Understanding the shape of the hazard rate: a process point of view. *Statistical Science*, 16: 1-22.
- Barlow R., and Proschan F. (1975). *Statistical Theory of Reliability and Life Testing. Probability Models*. New-York: Holt, Rinehart and Winston.
- Beard, R. E. (1959). Note on some mathematical mortality models. In: G. E. W. Woolstenholme and M. O'Connor (eds.). *The Lifespan of Animals*. Boston: Little, Brown and Company: 302-311.
- Beard, R. E. (1971). Some aspects of theories of mortality, cause of death analysis, forecasting and stochastic processes. In: W. Brass (ed.). *Biological aspects of demography*. London: Taylor & Francis: 57-68.
- Finkelstein M. S. (2000). Modeling a process of non-ideal repair. In: *Recent Advances in Reliability Theory*. N. Limnios and M. Nikulin. (eds). Birkhauser: 41-53.
- Finkelstein, M. S., and Esaulova V. (2001). Modeling a failure rate for a mixture of distribution functions. *Probability in the Engineering and Informational Sciences*, 15: 383-400.
- Finkelstein, M. S. (2002). On the shape of the mean residual life function. *Applied Stochastic Models in Business and Industry*, 18: 135-146.
- Finkelstein, M. S. (2003). A model of biological aging and the shape of the observed hazard rate. *Lifetime Data Analysis*, 9: 93-109.
- Gavrilov, N. A., and Gavrilova N. S. (1991). *The Biology of Life Span: A Quantitative Approach*: Harwood Academic Publishers.
- Gavrilov, N. A., and Gavrilova N. S. (2001). The reliability theory of aging and longevity. *Journal of Theoretical Biology*, 213: 527-545.
- Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society*, 115: 513-585.
- Gupta, R. C., and Akman, H. O. (1995). Mean residual life function for certain types of non-monotonic aging. *Stochastic models*, 11: 219-225.
- Horiuchi, S. (2003). Interspecies differences in the life span distribution: humans versus invertebrates. In: J. K. Carey and S. Tuljapurka (eds). *Life Span. Evolutionary*,

- Ecological and Demographic Perspectives. A Supplement to vol.29: Population and Development Review*: 127-151.
- Kirkwood, T. B. (1997). The origins of human aging. *Philosophical Transactions of the Royal Society of London-Series B: Biological Sciences*, 352(1363): 1765-1772.
- Lynn, N. J., and Singpurwalla, N.D. (1997). Comment: “Burn-in” makes us feel good. *Statistical Science*, 12: 13-19.
- Makeham, W. M. (1867). On the law of mortality. *Journal of the Institute of Actuaries*, 13: 325-358.
- Shaked, M., and Shantikumar J. (1993). *Stochastic Orders and Their Applications*. Boston: Academic Press,
- Shaked, M., and Spizzichino, F. (2001). Mixtures and monotonicity of failure rate functions In: N. Balakrishnan and C. R. Rao, eds. *Handbook of Statistics*, 20: 185-198, London: Elsevier,.
- Steinsaltz, D., and Evans, S. (2004). Markov mortality models: Implications of quasistationarity and varying initial distributions. *Theoretical Population Biology*, 65:4 (June).
- Thatcher, A. R. (1999). The long-term pattern of adult mortality and the highest attained age *J. R. Statist. Soc. A*, 162: 5-43.
- Vaupel, J. W., Manton K. G., and Stallard E. (1979). The impact of heterogeneity in individual frailty on the dynamics of mortality. *Demography*, 16: 439-454.
- Vaupel, J. W. (1998). Demographic analysis of aging and longevity. *American Economic Review*, 88:2, 242-247.
- Vaupel, J. W. (2003). Post-Darwinian longevity. In: J. K. Carey and S. Tuljapurkar (eds). *Life Span. Evolutionary, Ecological and Demographic Perspectives. A Supplement to vol.29: Population and Development Review*: 127-151.
- Wachter, K.W. (2003). Hazard curves and life span prospects. In: J. K. Carey and S. Tuljapurkar (eds). *Life Span. Evolutionary, Ecological and Demographic Perspectives. A Supplement to vol.29: Population and Development Review*: 270-291.
- Weitz, J., and Frazer, H. (2001). Explaining mortality rates plateaus. *Proceedings of the National Academy of Sciences, USA*, 98: 15383-15386.

- Yashin, A. I., Vaupel, J. W., and Iachine, I.A. (1994). A duality of aging: the equivalence of mortality models based on radically different concepts. *Mechanisms of Aging and Development*, 74: 1-14.
- Yashin, A. I., and Manton, K. G. (1997). Effects of unobserved and partially observed covariate processes on system failure: a review of models and estimation strategies. *Statistical Science*, 12: 20-34.
- Yashin, A. I., Iachine, I.A., and Begun, A. S. (2000). Mortality modeling: a review. *Mathematical Population Studies*, 8: 305-332.