DEMOGRAPHIC RESEARCH

Research Article

Increments to life and mortality tempo

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Griffith Feeney [1]

Abstract

This paper introduces and develops the idea of "increments to life." Increments to life are roughly analogous to forces of mortality: they are quantities specified for each age and time by a mathematical function of two variables that may be used to describe, analyze and model changing length of life in populations.

The rationale is three-fold. First, I wanted a general mathematical representation of Bongaart's "life extension" pill (Bongaarts and Feeney 2003) allowing for continuous variation in age and time. This is accomplished in sections 3-5, to which sections 1-2 are preliminaries. It turned out to be a good deal more difficult than I expected, partly on account of the mathematics, but mostly because it requires thinking in very unaccustomed ways.

Second, I wanted a means of assessing the robustness of the Bongaarts-Feeney mortality tempo adjustment formula (Bongaarts and Feeney 2003) against variations in increments to life by age. Section 6 shows how the increments to life mathematics accomplishes this with an application to the Swedish data used in Bongaarts and Feeney (2003). In this application, at least, the Bongaarts-Feeney adjustment is robust.

Third, I hoped by formulating age-variable increments to life to avoid the slight awkwardness of working with conditional rather than unconditional survival functions. This third aim has not been accomplished, but this appears to be because it was unreasonable to begin with. While it is possible to conceptualize length of life as completely described by an age-varying increments to life function, this is not consistent with the Bongaarts-Feeney mortality tempo adjustment.

What seems to be needed, rather, is a model that incorporates two fundamentally different kinds of changes in mortality and length of life, one based on the familiar force of mortality function, the other based on the increments to life function. Section 7 considers heuristically what such models might look like.

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1. Time-discrete increments to life

Figure 1 shows cohort survival for two birth cohorts of Swedish females. In the usual way of thinking, the survival curve for the later cohort has moved up because risks of death have declined, but we might equally well think of the curve for the later cohort as having moved to the right as a result of the prolongation of life.

To quantify this idea, consider the earlier cohort, choose a particular age ($x = 50$ years, say) and consider the horizontal distance between the two survival curves at the corresponding survival proportion, $\ell_c(50, t_1) = 0.6666$ (Figure 1), t_1 denoting the time of birth of the earlier cohort. To calculate this distance we need to know the age to which this proportion of persons survive in the later cohort. Interpolating on the values for the later cohort we find this age to be 60.65 years, i.e., $\ell(60.65, t) = 0.6666$. The horizontal distance between the two curves at the ordinate value $\ell_c(50, t_1) = \ell_c(60.65, t_2) = 0.6666$ is thus $\lambda_c^{t_1, t_2}(50) = 10.65$ years.

The difference between any two survival curves may be described as the collection of all such horizontal distances. These "increments to life" are plotted in Figure 2. The increment for any given age represents "how much longer" persons in the second cohort live in a rather special and formal sense. The persons in the second cohort who survive to age $x + \lambda_c^{t_1, t_2}(x)$ live $\lambda_c^{t, n}(x)$ years longer than the persons in the first cohort who survive to age x. Their advantage is retrospective, however, not prospective. The increment to life for older ages may be smaller, zero or negative.

The area under the increments to life curve is the difference between the areas under the survival curves. Since the area under the survival curves gives the expectation of life at birth for the two cohorts, we have the following decomposition of the difference between the expectations of life at birth in the two cohorts in terms of the increments to life values,

$$e_0^c(t_2) - e_0^c(t_1) = -\int_0^\infty \lambda_c^{t_1, t_2}(x) \, d\ell_c(x, t_1) \qquad (1)$$

where the integral is taken with respect to the first survivorship function.

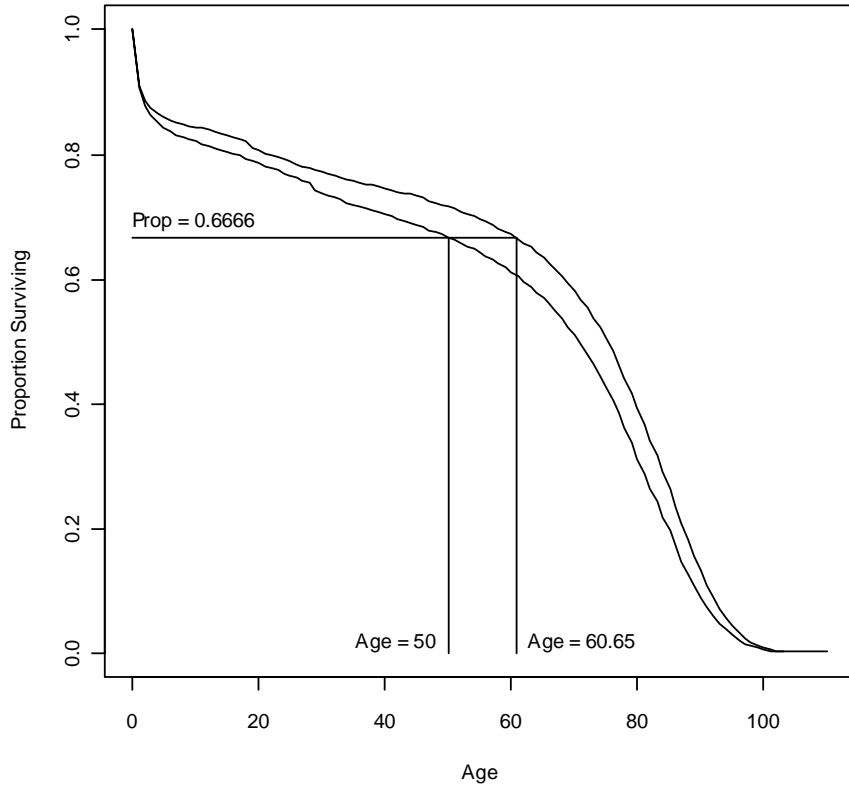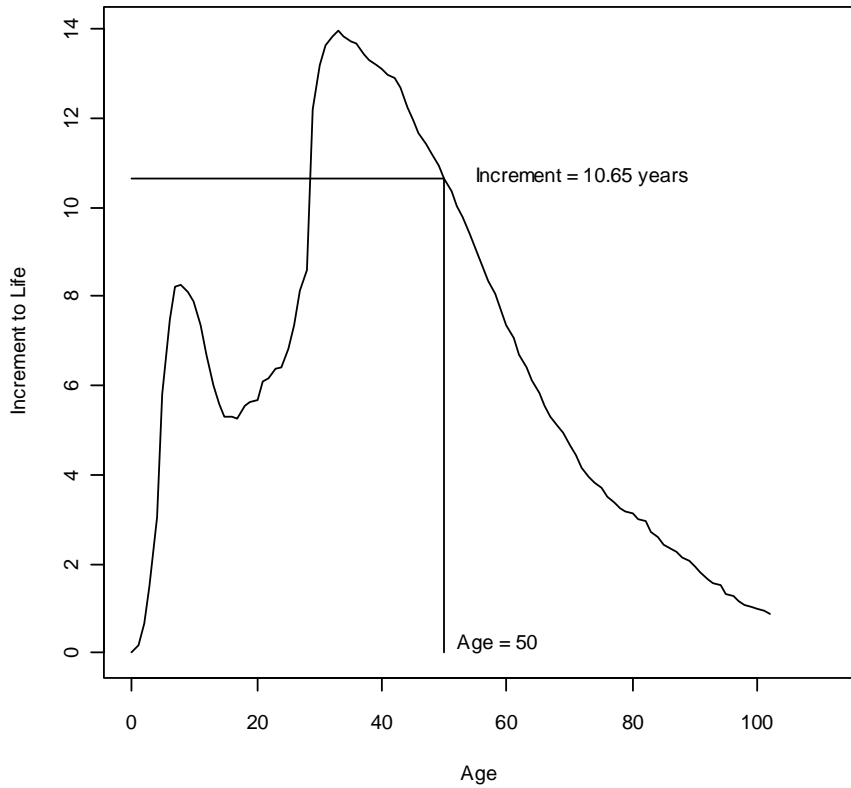**Figure 1:**   **Survivorship for Swedish Female Cohorts of 1890 and 1900**

Figure 2: **Time Discrete Increments to Life for Swedish Female Cohorts of 1890 and 1900**

2. Empirical results: Swedish females, 1751-2002

Increments to life by single years of age may be calculated for successive pairs of annual birth cohorts for Swedish females using the data provided in the Human Mortality Database (http://www.mortality.org). The database provides period life tables by single years of age to age 110 years for Sweden for (as of September 2004) 252 years, from 1751 through 2002. The q_x values from these tables may be used to compute cumulative cohort survival for the birth cohorts of persons born at the beginning of each calendar year. Applying the calculation of the preceding section to each successive pair of cohorts gives increments to life by single years of age for successive pairs of cohorts. These values may be arranged in a table in which rows correspond to single years of age and columns to pairs of adjacent birth cohorts and therefore to calendar years.

Figure 3 shows increments to life averaged over successive pairs of birth cohorts for the period 1751-1760. It illustrates that increments to life may be negative as well as positive, corresponding to a rise in mortality risks and a decline in length of life. Figure 4 shows increments to life averaged over successive pairs of birth cohorts for the period 1891-1900. Values are positive here, and the age pattern quite different. The depression at young adult ages is notable.

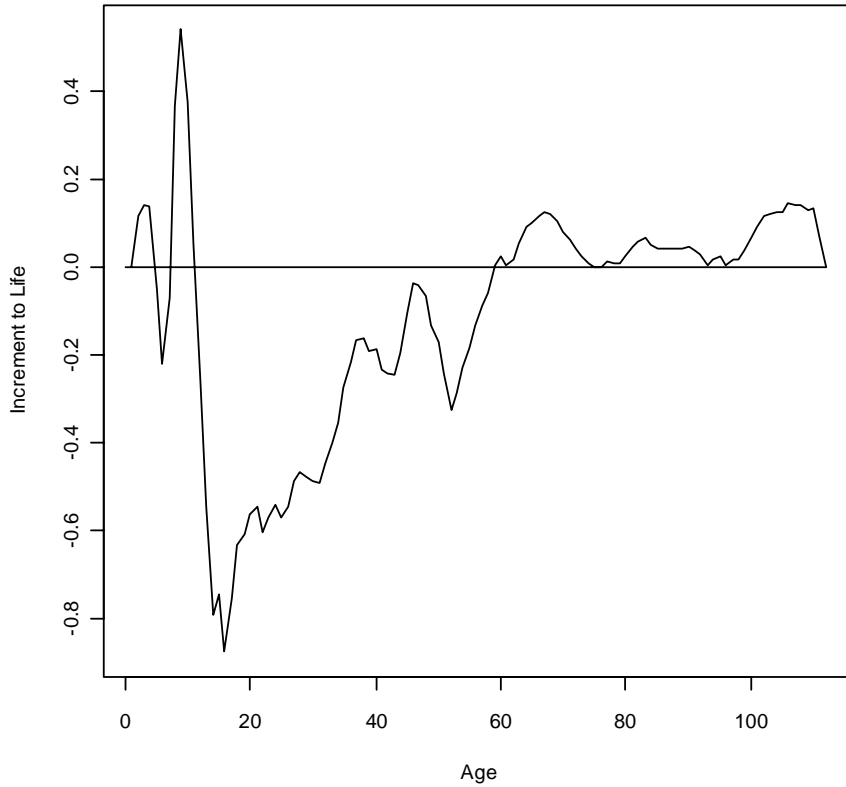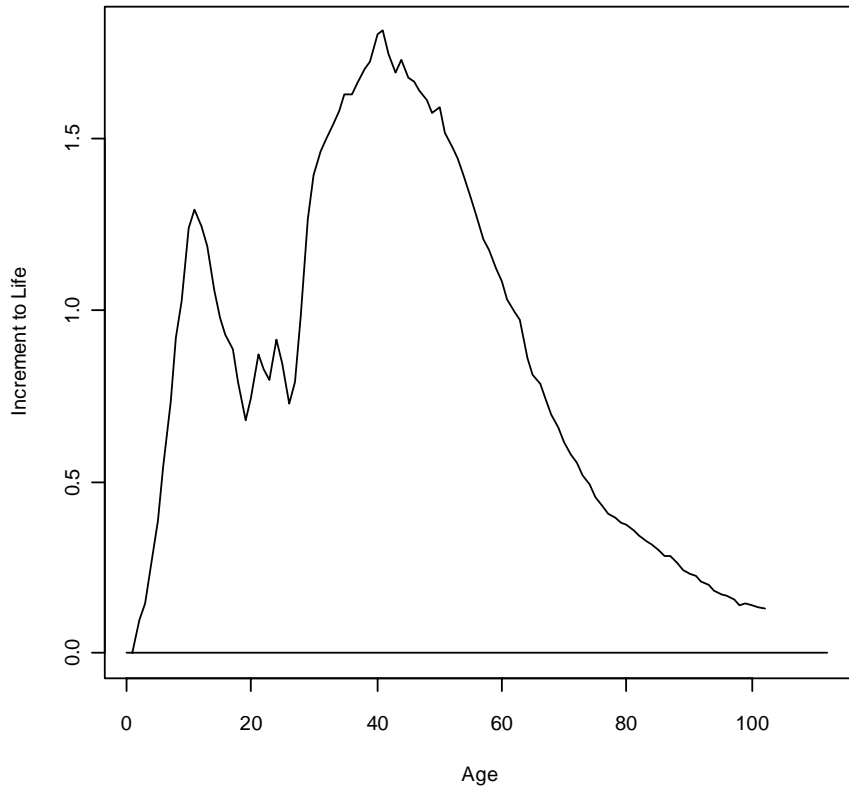**Figure 3:** **Time-Continuous Cohort Increments to Life, Swedish Females, Average over Cohorts of 1751-1760**

Figure 4:　　　**Time-Continuous Cohort Increments to Life, Swedish Females,**
　　　　　　　　Average over Cohorts of 1891-1900

3. Time-continuous cohort-indexed increments to life

Let $\ell_c(x,t)$ denote the proportion of persons surviving to age x in the cohort of persons born at time t. These values define a two-dimensional surface over the age-time plane of the Lexis diagram. This surface may be described by its contour lines, the lines on the age-time plane along which proportions surviving are constant. If length of life is constant, these contour lines will be straight lines parallel to the time axis. If length of life is increasing (decreasing), they will move to higher (lower) ages. The assumption that the population age distribution defined by $\ell_c(x,t)$ shifts to uniformly to higher ages (Bongaarts and Feeney 2002:16) is equivalent to the assumption that the rate of change of the contour lines with respect to age at any given time is invariant with respect to age.

Let the rate of change with respect to age of the contour line passing through the point (x,t) be $\lambda(x,t)$. The directional derivative of the surface defined by $\ell_c(x,t)$ in the direction $(\lambda(x,t),1)$ equals zero because the value of $\ell_c(x,t)$ does not change on the contour line. We therefore have

$$\frac{\partial \ell_c(x,t)}{\partial x}\lambda_c(x,t) + \frac{\partial \ell_c(x,t)}{\partial t} = 0 , \tag{2}$$

where the constant factor in the definition of the directional derivative may be ignored since the value is zero. Formula (2) is equivalent to

$$\lambda_c(x,t) = -\left[\frac{\partial \ell_c(x,t)/\partial t}{\partial \ell_c(x,t)/\partial x}\right], \tag{3}$$

which may be taken as the formal definition of the time-continuous cohort-indexed increment to life $\lambda_c(x,t)$ at age x and time t. The partial derivative in the denominator shows that empirical increments to life values will tend to be unstable over age intervals over which few deaths occur, since for these intervals $\partial \ell_c(x,t)/\partial x$ will be close to zero.

Dividing both sides of (2) by $\ell_c(x,t)$ and rearranging terms gives

$$\lambda_c(x,t)\mu(x,t) = r(x,t+x) , \tag{4}$$

where $\mu(x,t)$ denotes the force of mortality at age x and time t and $r(x,t)$ denotes the age-specific growth rate at age x and time t of the normalized population $\ell_c(\bullet,\bullet)$. This shows that values of the increments to life function vary inversely with the values of the force of mortality function for any given age and time.

The definition of increments to life by formula (3) supposes that the values $\ell_c(x,t)$ are given. If we assume instead that values $\lambda_c(x,t)$ are given, formula (2) defines a partial differential equation that may be solved for the values $\ell_c(x,t)$ given the boundary condition $\ell_c(x,0)$ for $x > 0$.

4. Time-continuous period-indexed increments to life

Let $\ell_p(x,t)$ denote the proportion of persons born at time $t - x$ who survive to age x. From this definition and that of $\ell_c(x,t)$ it follows immediately that

$$\ell_p(x,t) = \ell_c(x,t-x) \text{ and} \tag{5a}$$

$$\ell_c(x,t) = \ell_p(x,t+x). \tag{5b}$$

Compare Appendix 1 of Bongaarts and Feeney (1998), which states the same relation using slightly different notation. The subscripts refer to the cohort indexing of the preceding section and the period indexing of this section. Note that both $\ell_p(x,t)$ and $\ell_c(x,t)$ are survival proportions for cohorts; the difference is only in the time reference.

The apparently trifling difference between the two representations turns out to have non-trivial consequences. Proceeding as before, consider contour lines of the surface defined by the values $\ell_p(x,t)$. In the period case these contour lines may move backward as well as forward in time. Backward movement will occur whenever a later cohort experiences much lower survivorship than an earlier cohort.

Suppose for example that (a) for the cohort born at time t, half of all persons survive to age 50 years, corresponding to the point $(50, t+50)$ and that (b) the cohort born at time $t+1$ experiences much higher infant mortality, with the result that the age to which half of all persons in the cohort survive is only 40 years, corresponding to the point $(40, t+41)$. The time coordinate of the point for the later cohort lies 9 years before the time coordinate of the point for the earlier cohort.

The time-continuous increment to life may still be defined as the direction for which the directional derivative equals zero, but this direction must now be specified as a vector rather than as a scalar. The period version of formula (2) is

$$\frac{\partial \ell_p(x,t)}{\partial x} \lambda_p^1(x,t) + \frac{\partial \ell_p(x,t)}{\partial t} \lambda_p^2(x,t) = 0 \,, \tag{6}$$

where the vector $(\lambda_p^1(x,t), \lambda_p^2(x,t))$ gives the direction of the tangent to the contour line at the point (x,t). For consistency with the cohort formulation we may assume that $\lambda_p^2(x,t)$ assumes only the values $+1$ and -1, corresponding to movement forward and backward in time.

## 5. Relation between cohort and period increments to life

Figure 5 shows a Lexis diagram in which the diagonal line beginning at time t and ending at time $t+1+\lambda_c$ represents the tangent line to the contour line that passes through the point (x,t) of the surface $\ell_p(\bullet,\bullet)$. The slope of this line is by definition the period increment to life $\lambda_p = \lambda_p(x,t)$.

The corresponding rate of change between the cohorts born at times $t-x$ and $t-x+1$, represented by the dotted diagonal lines, is $\lambda_c = \lambda_c(x,t-x)$. From the similarity of the two right triangles,

$$\lambda_p(x,t-x) = \frac{\lambda_c(x,t-x)}{1+\lambda_c(x,t-x)} \,, \tag{7}$$

from which it follows that $\lambda_p(x,t) \to 1$ as $\lambda_c(x,t) \to \infty$ and $\lambda_p(x,t) \to -\infty$ as $\lambda_c(x,t) \to -1$. Values of $\lambda_c(x,t)$ less than -1 correspond contour lines moving backward in time.

Figure 5: **Lexis Diagram Illustrating Relation Between Cohort and Period Increment to Life**
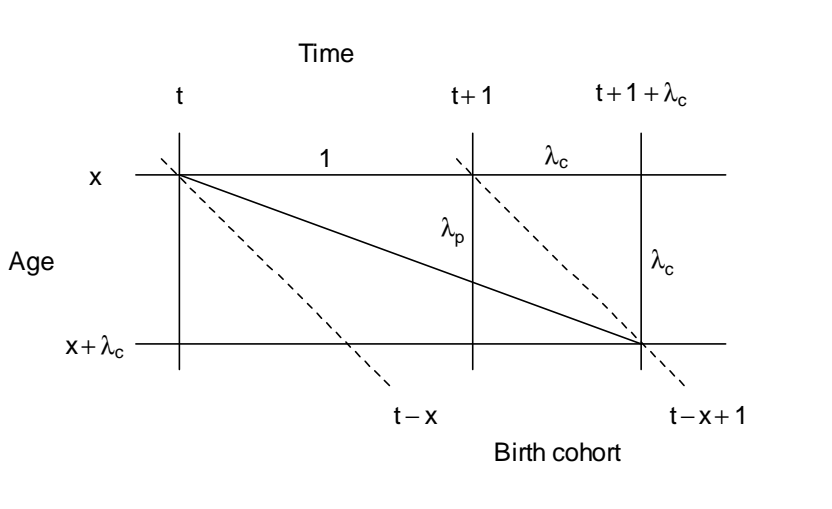
Zeng Yi and Land (2002) prove a special case of (7) for a model in which cohort fertility, period fertility, the shape of the age-schedule of fertility and the rate of change in the mean age at childbearing are all constant over time.

To obtain a more general formula, observe that the partial derivatives in (6) may be expressed as

$$\frac{\partial \ell_p(x,t)}{\partial x} = \frac{\partial \ell_c(x,t-x)}{\partial x} - \frac{\partial \ell_c(x,t-x)}{\partial t} \tag{8a}$$

and

$$\frac{\partial \ell_p(x,t)}{\partial t} = \frac{\partial \ell_c(x,t-x)}{\partial t}, \tag{8b}$$

these expressions being obtained by differentiating (5a). Substituting the right hand sides here in (6) and rearranging terms gives

$$\lambda_p^1(x,t) = \frac{-\partial \ell_c(x,t-x)/\partial t}{\partial \ell_c(x,t-x)/\partial x - \partial \ell_c(x,t-x)/\partial t} \tag{9a}$$

if $\lambda_p^2(x,t) = +1$ and

$$\lambda_p^1(x,t) = \frac{\partial \ell_c(x,t-x)/\partial x}{\partial \ell_c(x,t-x)/\partial x - \partial \ell_c(x,t-x)/\partial t} \tag{9b}$$

if $\lambda_p^2(x,t) = -1$. Dividing the numerator and denominator on the right hand sides of (9) gives

$$\lambda_p^1(x,t) = \frac{\lambda_c(x,t-x)}{1+\lambda_c(x,t-x)}, \quad \lambda_c(x,t) > -1, \tag{10a}$$

when $\lambda_p^2(x,t) = +1$ and

$$\lambda_p^1(x,t) = \frac{-\lambda_c(x,t-x)}{1+\lambda_c(x,t-x)}, \quad \lambda_c(x,t) < -1 \tag{10b}$$

when $\lambda_p^2(x,t) = -1$. Formula (10a) is the same as formula (7), but the graphical approach leaves it unclear how to cope with the case in which $\lambda_p^2(x,t) = -1$ or, equivalently, $\lambda_c(x,t) < -1$.

The relationship between $\lambda_c(x,t)$, $\lambda_p^1(x,t)$ and $\lambda_p^2(x,t)$ is shown in Figure 6. The curve to the right of the vertical at $\lambda_c(x,t) = 1$ shows the relation between $\lambda_c(x,t)$ and $\lambda_p^1(x,t)$ when $\lambda_p^2(x,t) = +1$ and the curve to the left of this vertical shows this relation when $\lambda_p^2(x,t) = -1$.

The relation displayed in Figure 6 is curious indeed. Discussion of tempo effects in the demographic literature has generally (always, so far as I am aware) been limited to values of λ_c and λ_p fairly close to zero (roughly, say, the unit square centered on the origin), and in this neighborhood the relationship is unremarkable. The Lexis diagram in Figure 5 shows that λ_p cannot exceed one, whereas λ_c may assume arbitrarily large values, so it is not surprising to see in Figure 6 that $\lambda_p \to 1$ as $\lambda_c \to \infty$. To see

$\lambda_p \to -\infty$ as $\lambda_c \to -1$ is rather less comfortable (though obviously, from (10a), this is what happens), since this suggests that tempo effects in this case can have arbitrarily large magnitude. In demographic terms (Lexis diagram in Figure 5), events in successive cohorts are shifting to younger ages in such a way as to pile up events on the vertical line at time t.

The portion of Figure 6 to the left of the vertical (dotted line) at $x = -1$ is even more surprising. The idea that events occurring in successive cohorts may be moved to earlier ages so rapidly that the period effect is to "thin out" events and reduce period levels rather than to "bunching up" events and increase period levels has not, so far as I am aware, ever been considered in the demographic literature. Yet this is what happens when $\lambda_c < 1$. In demographic terms (Lexis diagram in Figure 5), events in subsequent cohorts are moved to earlier ages so rapidly that they occur earlier in time than events to earlier cohorts. The asymptotic approach to λ_p to the left of the vertical line (dotted) at $x = -1$ mirrors the asymptote on the other side, but with λ_c decelerating toward -1. Of course the value of λ_c is constrained on the left because events cannot be shifted to a time before the cohort's birth!

**Figure 6:** **Relation Between Cohort and Period Increments to Life**
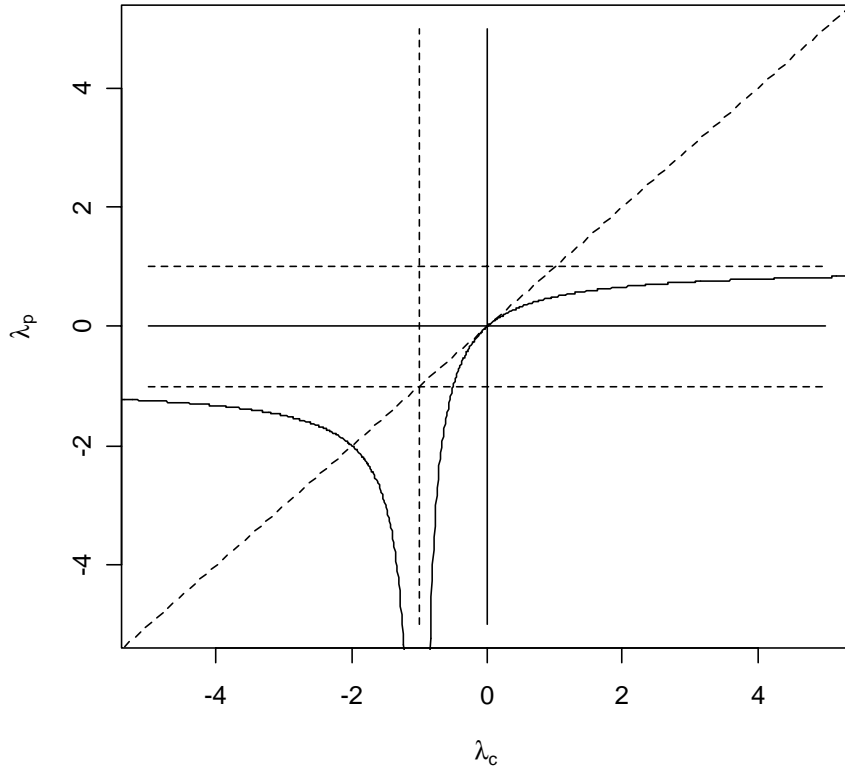
6. Robustness of the Bongaarts-Feeney Tempo Adjustment Formula

The Bongaarts-Feeney mortality tempo adjustment formula (Bongaarts and Feeney 2002, 2003) is based on the "constant shape assumption," which they show to be equivalent to the assumption that the normalized age distributions $\ell_p(x,t)$ are translated uniformly up or down the age axis with changing time. This is equivalent to the assumption that period increments to life $\lambda_p(x,t)$ are constant with respect to age for each time t, $\lambda_p(x,t) = \lambda(t)$ for all a. This suggests that tempo adjusted life expectancy at birth may be calculated more generally by replacing $\lambda(t)$ by $\lambda_p(x,t)$ in the Bongaarts-Feeney tempo adjustment formula (2003: formula 11, in which $\lambda(t) = \partial M_1(t)/\partial t$.

This adjustment may be applied to average of annual values of q_x for Swedish females for 1980-1995 with q_x set equal to zero for $x < 30$ years, the same Swedish data used in Bongaarts and Feeney (2003). Values of $\lambda_p(x,t)$ are obtained by first calculating $\lambda_c(x,t)$ using formula (3) and then applying formula (10) to obtain values of $\lambda_p(x,t)$. The resulting period increments to life by age $\lambda_p(x,t)$ are plotted in Figure 7, which suggests that they are reasonably close to constant with respect to age from about age 35 onward.

Calculation of a tempo-adjusted e_0 using these values gives 79.5 years, as compared with an unadjusted value of $e_0 = 81.0$ years, for a tempo effect of 1.5 years. This is very close to the 1.6 years given in Bongaarts and Feeney (2003). I conclude that the simple, non-age-specific adjustment is robust against observed departures from the constant shape assumption in this application, and also that the increments to life concept has succeeded in providing a general method for assessing robustness.

Figure 7: **Time-Continuous Period Increments to Life, Swedish Females, 1980-1995 (qx = 0 for x < 30 years)**
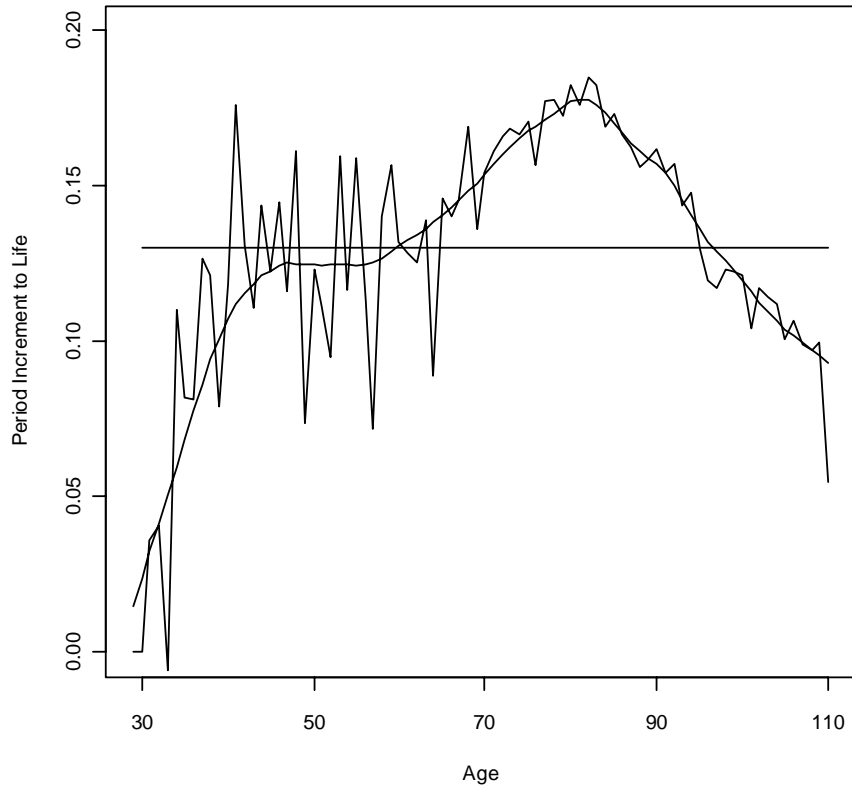
7. Increments to life and mortality tempo: mixed models

What happens if the conditioning on survival to mid-adult ages is dropped and variable increments to life are substituted for the constant increment to life used in the Bongaarts-Feeney adjustment formula? The procedure described in the previous section gives in this case an expectation of life more than 5 years lower than the conventional expectation of life. The magnitude of the implied tempo effects is about three times larger than the tempo effects calculated by Bongaarts and Feeney.

The explanation for this discrepancy is evidently the age variation in increments to life shown Figures 3 and 4. The Bongaarts-Feeney mortality tempo adjustment is derived on the assumption that increments to life are constant with respect to age. When the survival function is conditional on survival to age 30 years, the Swedish increments to life 1980-1995 vary in a range of about ± 0.05, as shown in Figure 7. When the survival function is unconditional, increments are very far from constant. Figure 4 shows a variation of about ± 0.9. Conditioning on survival to age 30 has the effect of radically reducing the variability of increments to life by age.

Consistency with the Bongaarts-Feeney mortality tempo model therefore requires that increments to life be considered only for adult survival. The nature of mortality change at younger and older ages appears to be fundamentally different, so that the tempo model that makes sense at older ages does not make sense at younger ages.

This suggests that we need a "mixed" model in which mortality change at younger ages is modeled differently from mortality change at older ages. To suggest what such models might look like, consider the familiar graph of the force of mortality function with values (vertical axis) plotted against age (horizontal axis). Thinking heuristically, suppose that there are two kinds of mortality change, "up and down" change (movement in the vertical direction to higher or lower values), and "back and forth" change (movement of a fixed schedule of values in the horizontal direction, to the left or to the right). Suppose further than "up and down" change occurs in infancy, childhood and young adult ages, and that "back and forth" change occurs at older ages.

The force of mortality function may be most appropriate representation of "up and down" change, the increments to life function the most appropriate representation of "back and forth" change. The distinction may be captured mathematically by writing the Makeham force of mortality function as $\mu(x,t) = a(t)\exp(bx) + c(t)$, where $c(t)$ represents "up and down" change and $a(t)$ represents "back and forth" change that may be equivalently expressed in terms of increment to life values $\lambda(t)$ representing the rate at which movement toward older or younger ages occurs.

So regarded, the Makeham defines a mixed model incorporating both forces of mortality and increments to life. Both components of the model could be generalized, to arrive at a more realistic model without changing the mixed nature of the model.

8. Conclusion

The study of mortality and length of life has been dominated by the concept of risks of death, to the point that mortality is sometimes regarded as being *defined* by age-specific death rates and the force of mortality function. Empirically, however, survival functions are the theoretical structure closest to the empirical data (migration may be handled with product limit survival functions), and changing survival functions give rise to and may be modeled by both forces of mortality and increments to life.

When we think in terms of risks of death, life times are a residual. How long we live reflects how successful we are in escaping various risks of death. When we think in terms of increments to life, deaths are the residual. Death is what happens when we run out of life. As pointed out by Vaupel and Yashin (1987), physicians and health personnel tend to think more in the latter terms than the former. They suggest also that the two perspectives are complementary rather than contradictory. A better understanding of this complementarity may usefully advance the study of changing mortality and length of life.

9. Acknowledgements

http://www.demographic-research.org

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