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Formal Relationships 18

How life expectancy varies with perturbations in age-specific mortality

Tomasz F. Wrycza

Annette Baudisch

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How life expectancy varies with perturbations in age-specific mortality

Tomasz F. Wrycza[1, 3]

Annette Baudisch[2, 3]

Abstract

BACKGROUND
A naturally arising question in demography is how a given change in mortality affects life expectancy. Scholars have targeted this question with different aims and from different perspectives.

OBJECTIVE
We present and prove the central relationship between change in mortality and resulting change in e_0, and we systematically apply it to investigate the effect of specific mortality perturbations.

COMMENTS
Expressions for the change in e_0 resulting from a change in the parameters of the standard parametric mortality model in demography, the Gompertz–Makeham model, include well-known demographic quantities, which might prove useful for future studies.

[1] Max Planck Institute for Demographic Research, Germany. E-mail: wrycza@demogr.mpg.de.
[2] Max Planck Institute for Demographic Research, Germany. E-mail: baudisch@demogr.mpg.de.
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1. Relationship

Life expectancy at age zero is defined as

$$e_0 \; = \; \int_0^\infty l(x)\, dx,$$

where $l(x)$ denotes the probability of survival to age x. Let $e(a)$ denote remaining life expectancy at age a, given by

$$e(a) \; = \; \frac{1}{l(a)} \int_a^\infty l(t)\, dt.$$

Assume that mortality $\mu(x)$ is perturbed, resulting in mortality $\mu(x,\epsilon)$ (where the parameter ϵ captures the perturbation), such that

$$\lim_{\epsilon \to 0} \mu(x,\epsilon) \; = \; \mu(x,0) \; = \; \mu(x) \;\; \forall x.$$

Using the perturbed life expectancy $e_0(\epsilon)$, one can define the derivative

$$\frac{de_0}{d\epsilon} \; = \; \lim_{\epsilon \to 0} \frac{e_0(\epsilon) - e_0}{\epsilon}.$$

The general relationship between a perturbation in mortality and the resulting perturbation in life-expectancy in terms of derivatives is then given by

$$\frac{de_0}{d\epsilon} \; = \; -\int_0^\infty \frac{\partial \mu}{\partial \epsilon}(x,0)\, e(x) l(x)\, dx. \tag{1}$$

This means that the change in life expectancy can be calculated by summing up the changes in mortality over all ages, weighted by the remaining person-years of expected life at each age.

2. Proof

Using first order Taylor approximation with $\mu(x,\epsilon)$, we get

$$\mu(x,\epsilon) \approx \mu(x) + \epsilon \frac{\partial \mu}{\partial \epsilon}(x,0)$$

and therefore

$$l(x,\epsilon) \approx e^{-\int_0^x \mu(t)dt - \epsilon \int_0^x \frac{\partial \mu}{\partial \epsilon}(t,0)dt} = l(x)e^{-\epsilon \int_0^x \frac{\partial \mu}{\partial \epsilon}(t,0)dt}.$$

Applying first order Taylor approximation to the exponential, this gives

$$l(x,\epsilon) \approx l(x)\left(1 - \epsilon \int_0^x \frac{\partial \mu}{\partial \epsilon}(t,0)dt\right).$$

Using this to calculate life expectancy, we arrive at

$$e_0(\epsilon) \approx e_0 - \epsilon \int_0^\infty l(x) \int_0^x \frac{\partial \mu}{\partial \epsilon}(t,0)dt\,dx.$$

Applying integration by parts, we get

$$e_0(\epsilon) \approx e_0 - \epsilon \int_0^\infty \frac{\partial \mu}{\partial \epsilon}(x,0)e(x)l(x)dx,$$

because

$$(e(x)l(x))' = -l(x) \text{ and } \left(\int_0^x \frac{\partial \mu}{\partial \epsilon}(t,0)dt\right)' = \frac{\partial \mu}{\partial \epsilon}(x,0).$$

This finally gives

$$\frac{de_0}{d\epsilon} = \lim_{\epsilon \to 0} \frac{e_0(\epsilon) - e_0}{\epsilon} = -\int_0^\infty \frac{\partial \mu}{\partial \epsilon}(x,0)\,e(x)l(x)\,dx,$$

which is precisely (1).
Q.E.D.

3. History and related results

A very early derivation of the presented relationship was given by Wilson (1938), although in a different formulation: the aim was to find an expression for the standard deviation of life expectancy as computed from life tables. Wilson assumes a small change (or error in this context) in q_x, the chance of dying between ages x and $x+h$, and computes the resulting change in life table e_0. Irwin (1949) derives a similar result, also in the context of statistical properties of life expectancy estimations from life tables.

Keyfitz (1971, 1977) derives the relationship as part of a general investigation into the effect of changes in birth and death rates on crucial stable population parameters.

Pollard (1982, 1988) provides a simple yet striking derivation of an exact formula for the change in life expectancy in terms of mortality changes, which also captures effects of higher orders. Expression (1) is easily derived from this result.

Arthur (1984) presents a general method based on functional differentials for deriving closed-form expressions for the sensitivity of demographic variables (in particular intrinsic growth rates) to changes in input functions or schedules. Although he does not explicitly mention the case of life expectancy, it is readily captured by his approach.

Vaupel (1986) provides an investigation into how the rate of progress in μ over time translates into the rate of progress in life expectancy. His focus is on the potential of life-saving efforts.

Focusing on improvement in e_0 over time, Vaupel and Canudas Romo (2003) use a formula equivalent to (1) to derive a decomposition of change in life expectancy consisting of a term capturing the general effect of a reduction in μ and a second term capturing heterogeneity in this reduction over age.

Wagner (2010), based on Zhang and Vaupel (2009), derives a similar relationship, but for the impact of a change in mortality on $e^\dagger = \int_0^\infty e(x)d(x)dx$ (where $d(x)$ is the probability density of age at death) rather than on e_0.

Goldstein and Cassidy (2012) investigate the effect of specific mortality changes (level changes and senescence changes, corresponding to Gompertz a and b respectively) and apply their results to human mortality data.

Caswell (2008, 2011a, 2011b) revisits the question of perturbation analysis from the standpoint of Markov chains and matrix population models, thus complementing the continuous approach discussed here.

4. Applications: Specific forms of perturbation

Equation (1) relates a change in mortality to a change in life expectancy. This basic relationship permits the derivation of several specific relationships between mortality and life expectancy under different perturbation regimes.

4.1 Systematic analysis of perturbations

In the following we investigate four general cases of mortality perturbation: additive, proportional, linearly growing and exponentially growing. For each case, we state how, specifically, mortality is perturbed and calculate the derivative $\frac{e_0}{d\epsilon}$ using (1) to analyze the effect of the perturbation on life expectancy. We use the subscripts 'add', 'prop', 'lin' and 'exp' for both the perturbed mortality $\mu(x, \epsilon)$ and the resulting derivatives to indicate the respective form of perturbation applied.

I) Additive:

$$\boxed{\mu_{\text{add}}(x,\epsilon) \;=\; \mu(x) \;+\; \epsilon, \quad \frac{d\mu}{d\epsilon}_{\text{add}}(x,0) \;=\; 1}$$

$$\frac{de_0}{d\epsilon}_{\text{add}} \;=\; -\int_0^\infty e(x)l(x)\,dx\,.$$

Acknowledging that according to Goldstein (2009)

$$\int_0^\infty e(x)l(x)\,dx \;=\; \int_0^\infty x\,l(x)\,dx,$$

it follows that

$$\frac{de_0}{d\epsilon}_{\text{add}} \;=\; -\overline{x}\,e_0. \tag{2}$$

The change in e_0 depends on two mean values: the mean age of the living (in the stationary population), denoted by $\overline{x}$, and life expectancy itself, which is equivalent to the mean age at death.

II) Proportional:

$$\boxed{\mu_{\text{prop}}(x,\epsilon) \;=\; (1 \;+\; \epsilon)\mu(x), \quad \frac{d\mu}{d\epsilon}_{\text{prop}}(x,0) \;=\; \mu(x)}$$

$$\begin{aligned}\frac{de_0}{d\epsilon}_{\text{prop}} &= -\int_0^\infty \mu(x)e(x)l(x)\,dx \\ &= -\int_0^\infty e(x)d(x)\,dx = \;-e^\dagger.\end{aligned} \tag{3}$$

If the intensity of a perturbation at each age is proportional to the level of mortality at this age, we find that the resulting life expectancy differential is closely connected to life disparity, which is measured by the average amount of life lost due to death, denoted as $e^\dagger$ (see e.g. Zhang and Vaupel (2009)).

III) Linearly growing:

$$\boxed{\mu_{\text{lin}}(x,\epsilon) \;=\; \mu(x) \;+\; \epsilon x, \quad \frac{d\mu}{d\epsilon}_{\text{lin}}(x,0) \;=\; x}$$

$$\frac{de_0}{d\epsilon}_{\text{lin}} \;=\; -\int_0^\infty x\,e(x)l(x)\,dx\,. \tag{4}$$

We might also use a different formulation of this result: If we consider that for two demographic functions $u(x)$ and $v(x)$ with weighting function $w(x)$ the covariance is

$$\text{(5)} \qquad \mathrm{Cov}_w(u,v) = \frac{\int_0^\infty u(x)v(x)w(x)\,dx}{\int_0^\infty w(x)\,dx} - \frac{\int_0^\infty u(x)w(x)\,dx}{\int_0^\infty w(x)\,dx} \cdot \frac{\int_0^\infty v(x)w(x)\,dx}{\int_0^\infty w(x)\,dx},$$

we get

$$\frac{de_0}{d\epsilon}_{\text{lin}} = -e_0\,\mathrm{Cov}_l(x,e) - \frac{1}{e_0}\left(\int_0^\infty x\,l(x)\,dx\right)\left(\int_0^\infty e(x)l(x)dx\right),$$

and because of Goldstein (2009) this reduces to

$$\text{(6)} \qquad \frac{de_0}{d\epsilon}_{\text{lin}} = -e_0\,(\mathrm{Cov}_l(x,e) + \overline{x}^2).$$

Because a linearly growing perturbation, unlike an additive perturbation, affects mortality at each age differently (i.e. more strongly), the resulting change in e_0 comprises two terms: $\overline{x}^2$, which is an average characteristic of the mortality regime (and can also be found in the case of additive change), and $\mathrm{Cov}_l(x,e)$, which captures the regime's heterogeneity over age.

IV) Exponentially growing:

$$\boxed{\mu_{\text{exp}}(x,\epsilon) = e^{\epsilon x}\mu(x), \quad \frac{d\mu}{d\epsilon}_{\text{exp}}(x,0) = x\,\mu(x)}$$

$$\text{(7)} \qquad \frac{de_0}{d\epsilon}_{\text{exp}} = -\int_0^\infty x\,e(x)d(x)\,dx\,.$$

If we use (5) with $d(x)$ (the *pdf* of age at death) as weighting function, we get

$$\frac{de_0}{d\epsilon}_{\text{exp}} = -\left(\mathrm{Cov}_d(x,e) + \left(\int_0^\infty xd(x)dx\right)\left(\int_0^\infty e(x)d(x)dx\right)\right),$$

which means

$$\text{(8)} \qquad \frac{de_0}{d\epsilon}_{\text{exp}} = -\,(\mathrm{Cov}_d(x,e) + e_0e^\dagger)\,.$$

Notice that this case of an exponentially growing perturbation is equivalent to a perturbation of the form $\mu(x,\epsilon) = (1+\epsilon\, x)\,\mu(x)$, because $e^{\epsilon x} \approx 1+\epsilon\, x$. So the exponentially growing perturbation is, in a sense, a variant of a proportional perturbation, but one that affects each age differently, so again the resulting change in e_0 can be decomposed into two terms, capturing different aspects of the mortality regime.

4.2 Systematic analysis in the Gompertz-Makeham model

One of the central mortality models used in demography is the Gompertz–Makeham mortality model, given by

$$\mu(x) = ae^{bx} + c.$$

In the following we analyze the effect of changes in parameters a, b and c on life expectancy.

i) Change in the initial level of age-dependent mortality: $\boxed{a(\epsilon) = a+\epsilon}$
In this case it is helpful to note that $e^{bx} = \frac{1}{a}\,(\,\mu(x)-c\,)$. Since

$$\mu(x,\epsilon) = ae^{bx} + \epsilon\, e^{bx} + c = \mu(x) + \epsilon\, e^{bx} = \mu(x) + \epsilon\left(\frac{\mu(x)-c}{a}\right),$$

the mortality derivative is given by

$$\frac{d\mu}{d\epsilon}(x,0) = \frac{1}{a}\,(\,\mu(x)-c\,)\,,$$

and therefore for $a \neq 0$ it holds that

$$\frac{de_0}{da} = \frac{1}{a}\left(\frac{de_0}{d\epsilon}_{\text{prop}} - c\,\frac{de_0}{d\epsilon}_{\text{add}}\right) = \frac{1}{a}\,(\,c\,\overline{x}\,e_0 - e^{\dagger}\,). \tag{9}$$

The change in life expectancy due to a change in the level of age-dependent mortality of the Gompertz-Makeham model is a linear combination of a proportional and an additive change for the general, non-parametric mortality pattern.

ii) Change in the rate of increase of mortality: $\boxed{b(\epsilon) = b+\epsilon}$
Noting that $ae^{bx} = \mu(x)-c$ we find that

$$\mu(x,\epsilon) = ae^{bx}e^{\epsilon x} + c = (\,\mu(x)-c\,)\,e^{\epsilon x} + c\,,$$

which determines the change in mortality to be

$$\frac{d\mu}{d\epsilon}(x,0) \;=\; x\,(\,\mu(x)-c\,)\,. \tag{10}$$

However, in the case of Gompertz-Makeham we also have

$$\mu'(x) = bae^{bx} = b(\mu(x)-c) \quad \Rightarrow \quad \mu(x)-c = \frac{1}{b}\mu'$$

for $b \neq 0$. Plugging this into (10) gives

$$\frac{d\mu}{d\epsilon}(x,0) \;=\frac{1}{b}x\mu'(x)$$

and therefore

$$\frac{de_0}{db} \;= -\frac{1}{b}\int_0^\infty x\mu'(x)e(x)l(x)dx.$$

Applying integration by parts and using

$$(xe(x)l(x))' = (e(x)-x)l(x)$$

yields

$$\int_0^\infty x\mu'(x)e(x)l(x)dx = \int_0^\infty xd(x)dx - \int_0^\infty e(x)d(x)dx,$$

so that

$$\frac{de_0}{db} \;= \frac{1}{b}(e^\dagger - e_0). \tag{11}$$

This relationship is a special case of the result derived in Goldstein and Cassidy (2012), where a mortality change of the form $\mu(x,\epsilon) = \mu((1+\epsilon)x)$ is investigated, which in the case of Gompertz-Makeham corresponds to a proportional change in b.

iii) Change in the level of age-independent mortality: $\boxed{c(\epsilon) = c+\epsilon}$
Because c is merely an age-independent additive term, this case is captured by the additive case derived under I),

$$\frac{de_0}{dc} \;=\; \frac{de_0}{d\epsilon}_{\text{add}} \;=\; -\overline{x}\,e_0. \tag{12}$$

4.3 Concluding remarks

The above derivations provide a systematic overview of the way different general perturbations in mortality translate into different forms of change in life expectancy. The perturbations for the specific case of the Gompertz – Makeham mortality model turn out to include well-studied demographic measures such as $e^{\dagger}$ and $\overline{x}$. This tight connection to central measures of demography might prove useful in future applications.

For instance, since Gompertz parameter b is often taken to reflect the rate of aging within the community of Gerontologists and Biodemographers, relationship (11) may support research concerned with changes in the rate of aging (as laid out in Goldstein and Cassidy (2012)). Additive age-independent perturbations in mortality as captured by relationships (2) and (12) may support studying the effect of "extrinsic" or "environmental" mortality, which plays an important role in biological applications, and in particular in evolutionary theories of aging. The results might also be useful for interpreting outcomes of empirical work when investigating what kind of mortality change is induced by certain types of manipulation, such as changes in diet, temperature, or genetic manipulations, which for the experimenter become measurable as differences in life expectancy between treatment groups.

Also, these relationships could help to connect the classic formal demography framework with the recently suggested new theoretical framework of *Pace* and *Shape* (Baudisch (2011)). *Pace* in this context refers to how fast organisms go through their life cycle, while *Shape* refers to the quality and degree of the change over the life cycle. Life expectancy measures the *Pace* of life, thus the relationships derived above may constitute constructive building blocks for the new framework.

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