Response Letter to "Gamma-Gompertz life expectancy at birth", *Demographic Research*, 12 February 2013

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An error and a typo in the original manuscript

The original article contains an incomplete expression, equation (10): to be equal to remaining life expectancy e_x, the expression on the right-hand-side of (10) needs to be divided by the survival function $S(x)$ of the gamma-Gompertz model. In this sense, equation (10) in the original manuscript is incorrect, and readers should not use its right-hand side to calculate e_x. Here, I provide the correct version for e_x and present an alternative derivation method that is based on the same substitution $t = 1 - e^{-bx}$ (see the transition from (5) to (6) in the original article) as the one I use to derive expression (2) for life expectancy e_0. The latter is also used by Castellares et al. (2020). In this Letter, I also correct a typo on p. 266, third (unnumbered) equation from the bottom, in which the interval endpoints for calculating the primitive on the right-hand side are missing. Throughout this Letter, I will use the Roman format for equation numbering to make a clear distinction from the equations in the original article.

A correct expression for e_x in a gamma-Gompertz setting

For a population in which a survival function $S(x)$ characterizes individual lifetimes, remaining life expectancy at age x is given by

$$e_x = \frac{1}{S(x)} \int\limits_x^\infty S(y)dy\,. \tag{i}$$

The original article presents an equation for $\int\limits_x^\infty S(y)dy$ only, i.e., equation (10) should be re-written with its correct left-hand side:

$$\int\limits_x^\infty S(y)dy = \frac{1}{bk}\left(\frac{b\lambda}{a}e^{-bx}\right)^k {}_2F_1\left(k,k;k+1;\left(1-\frac{b\lambda}{a}\right)e^{-bx}\right). \tag{ii}$$

Applying the linear transformation formula for hypergeometric functions (Abramowitz and Stegun, 1965, p.559, 15.3.4 or 15.3.5), namely,

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right).$$

or

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-b}\,{}_2F_1\left(b,c-a;c;\frac{z}{z-1}\right).$$

for $a = k$, $b = k$, $c = k+1$, and $z = \left(1-\frac{b\lambda}{a}\right)e^{-bx}$, yields

$$\int\limits_x^\infty S(y)dy = \frac{1}{bk}\left(\frac{1-\left(1-\frac{b\lambda}{a}\right)e^{-bx}}{\frac{b\lambda}{a}e^{-bx}}\right)^{-k} {}_2F_1\left(k,1;k+1;\frac{1-\frac{b\lambda}{a}}{1-\frac{b\lambda}{a}+\frac{b\lambda}{a}e^{-bx}}\right). \tag{iii}$$

Note that the expression

$$\left(\frac{1-\left(1-\frac{b\lambda}{a}\right)e^{-bx}}{\frac{b\lambda}{a}e^{-bx}}\right)^{-k} = \left(1+\frac{a}{b\lambda}(e^{bx}-1)\right)^{-k}.$$

The right-hand side of the latter represents the gamma-Gompertz survival function. As a result, remaining life expectancy in a gamma-Gompertz setting equals

$$e_x = \frac{1}{bk}\,{}_2F_1\left(k, 1; k+1; \frac{1-\frac{b\lambda}{a}}{1-\frac{b\lambda}{a}+\frac{b\lambda}{a}e^{-bx}}\right). \tag{iv}$$

The same expression is reported in Castellares et al. (2020).

Expressing e_x by applying the derivation procedure for e_0

The original article misses the opportunity to derive an expression for e_x by applying the same substitution as the one used for e_0, namely $t = 1 - e^{-bx}$, under the integral. In fact, it takes a simple substitution $u = y - x$ to reduce the initlal integral to a form that resembles expression (5) in the original article

$$\int_x^\infty S(y)dy = \int_x^\infty \left(1 + \frac{a}{b\lambda}\left(e^{by}-1\right)\right)^{-k} dy = \int_0^\infty \left(1 + \frac{a}{b\lambda}\left(e^{bu}e^{bx}-1\right)\right)^{-k} du\,. \tag{v}$$

Then the aforementioned substitution $t = 1 - e^{-bu}$ leads to the integral representation of the Gaussian hypergeometric function:

$$\begin{aligned}
\int_x^\infty S(y)dy &= \int_0^\infty \left(1 + \frac{a}{b\lambda}\left(e^{bu}e^{bx}-1\right)\right)^{-k} du = \frac{1}{b}\int_0^\infty \left(1 + \frac{a}{b\lambda}\left(\frac{e^{bx}}{1-t}-1\right)\right)^{-k} \frac{1}{1-t}\,dt = \\
&= \frac{1}{b}\int_0^\infty (1-t)^{k-1}\left(1 + \frac{a}{b\lambda}\left(e^{bx}-1\right) - \left(1-\frac{a}{b\lambda}\right)t\right)^{-k} dt = \\
&= \frac{1}{b}\left(1 + \frac{a}{b\lambda}\left(e^{bx}-1\right)\right)^{-k}\int_0^\infty (1-t)^{k-1}\left(1 - \left(1-\frac{a}{b\lambda}\right)t\right)^{-k} dt\,.
\end{aligned}$$

From here, it takes only making the appropriate correspondence with (Abramowitz and Stegun, 1965, p.558, 13.3.1)

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,, \qquad \Re(c) > \Re(b) > 0 \quad \text{(vi)}
$$

to get (iv). Note that the gamma-Gompertz parameters are real positive numbers, and the condition in (vi) reduces to $c > b > 0$, which is fulfilled.

A note on deriving (10) in the original article

The original article contains typos that were unfortunately overlooked in the proofs. In particular, the third equation from the bottom on p. 266 contains such a typo (the interval endpoints for calculating the primitive are missing) that makes the derivation of (10) look somewhat irrational. I also started deriving from the indefinite integral which is unnecessary. Here, I present the derivation once again more neatly. It is less elegant than the one in the previous section, but still holds.

When deriving (10), the original article uses a substitution $t = e^{-y}$. It leads to

$$
\int\limits_x^\infty S(y)dy = \int\limits_x^\infty \left(1 + \frac{a}{b\lambda}\left(e^{by}-1\right)\right)^{-k} dy = \frac{1}{b}\left(\frac{b\lambda}{a}\right)^k \int\limits_0^{e^{-bx}} t^{k-1}\left(1 - \left(1 - \frac{b\lambda}{a}\right) t\right)^{-k} dt \quad \text{(vii)}
$$

The interchangeability of the first two arguments of the hypergeometric function, i.e., ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$ and the relationship (Lebedev, 1965, p. 258)

$$
\left(1 - \left(1 - \frac{b\lambda}{a}\right) t\right)^{-k} = {}_2F_1\left(k, C; C; \left(1 - \frac{b\lambda}{a}\right) t\right) \qquad \forall C \equiv const \quad \text{(viii)}
$$

yield

$$
\int\limits_x^\infty S(y)dy = \frac{1}{b}\left(\frac{b\lambda}{a}\right)^k \int\limits_0^{e^{-bx}} t^{k-1}\,{}_2F_1\left(k,k;k;\left(1 - \frac{b\lambda}{a}\right) t\right) dt\,. \quad \text{(ix)}
$$

In addition, the relationship (see https://functions.wolfram.com/07.23.21.0006.01)

$$\int z^{c-1}\,{}_2F_1(a,b;c;z)\,dz = z^c\,\Gamma(c)\,{}_2\tilde{F}_1(a,b;c+1;z) = \frac{z^c}{c}\,{}_2F_1(a,b;c+1;z)\,, \tag{x}$$

where ${}_2\tilde{F}_1(a,b;c+1;z) = \frac{1}{\Gamma(c+1)}\,{}_2F_1(a,b;c+1;z)$ is the regularized hypergeometric function, implies

$$\int_x^\infty S(y)dy = \frac{1}{bk}\left(\frac{b\lambda}{a}\right)^k t^k\,{}_2F_1\left(k,k;k+1;\left(1-\frac{b\lambda}{a}\right)t\right)\Bigg|_0^{e^{-bx}}\,. \tag{xi}$$

By the definition of the Gaussian hypergeometric function, i.e.,

$${}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty}\frac{a(a+1)\ldots(a-j+1)\,b(b+1)\ldots(b-j+1)}{c(c+1)\ldots(c-j+1)\,j!}\,z^j\,, \tag{xii}$$

we have

$${}_2F_1\left(k,k;k+1;\left(1-\frac{b\lambda}{a}\right)t\right) = \sum_{j=0}^{\infty} C_j t^j\,, \tag{xiii}$$

where $C_j = \frac{k\cdot k(k+1)\ldots(k-j+1)}{(k-j+2)\,j!}\left(1-\frac{b\lambda}{a}\right)^j$.
As $k>0$,

$$\lim_{t\to 0} t^k\cdot{}_2F_1\left(k,k;k+1;\left(1-\frac{b\lambda}{a}\right)t\right) = \lim_{t\to 0}\sum_{j=0}^{\infty} C_j t^{j+k} = 0\,, \tag{xiv}$$

and (xi) reduces to

$$\int_x^\infty S(y)dy \;=\; \frac{1}{bk}\left(\frac{b\lambda}{a}e^{-bx}\right)^k\,{}_2F_1\left(k,k;k+1;\left(1-\frac{b\lambda}{a}\right)e^{-bx}\right)\,. \tag{xv}$$

The expression in the right-hand side of (xv) is equivalent to (10) in the original manuscript.

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