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Formal Relationship 21

Entropy of the Gompertz-Makeham mortality model

Tomasz F. Wrycza

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Entropy of the Gompertz-Makeham mortality model

Tomasz F. Wrycza [1]

Abstract

BACKGROUND
Life table entropy is a quantity frequently used in demography; e.g., as a measure of heterogeneity in age at death, or as the elasticity of life expectancy with regards to proportional changes in age-specific mortality. It is therefore instructive to calculate its value for the widely used Gompertz-Makeham mortality model.

OBJECTIVE
I present and prove a simple expression of life table entropy for the Gompertz-Makeham model, which ties together the parameters of the model with demographically relevant quantities.

COMMENTS
The relationship shows that entropy is easily calculated from the parameters of the given model, life expectancy e_0 and the average age in the stationary population $\bar{x}$. The latter enters the equation only if the Makeham term c is different from zero.

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1. Relationship

Let

$$e_0 = \frac{1}{l(0)} \int_0^\infty l(x)dx = \int_0^\infty l(x)dx$$

denote life expectancy at birth. Let $\bar{H}$ denote life table entropy. The notation $\bar{H}$ was suggested by J.W. Vaupel (personal communication) because $\bar{H}$ can be viewed as the average value of the cumulative hazard function $H(x) = \int_0^x \mu(t)dt$:

$$\bar{H} = \frac{\int_0^\infty H(x)l(x)dx}{\int_0^\infty l(x)dx}.$$

This expression for $\bar{H}$ follows from Keyfitz's (1977a) definition of the entropy of the life table as

$$\bar{H} = \frac{-\int_0^\infty \log(l(x))l(x)dx}{\int_0^\infty l(x)dx}.$$

The numerator $\int_0^\infty H(x)l(x)dx$ equals the average number of life-years lost due to death $e^\dagger$.

For the Gompertz-Makeham model of age-specific mortality

$$\mu(x) = ae^{bx} + c,$$

it holds that

$$\bar{H} = \frac{1}{b}\left(\frac{1}{e_0} - (a+c)\right) + c\bar{x} = \frac{\bar{\mu} - \mu_0}{b} + c\bar{x} \tag{1}$$

where $\bar{\mu} = \frac{1}{e_0}$ is the average, or crude, death rate in the stationary population,

$$\bar{x} = \frac{1}{e_0}\int_0^\infty xl(x)dx$$

is the average age in the stationary population, and $\mu_0 = a + c$ is the initial mortality.

2. Proof

Let $\mu(x) = ae^{bx} + c$. Then

$$H(x) = \int_0^x \mu(t)dt = \frac{1}{b}(ae^{bx} - a) + cx = \frac{1}{b}(\mu(x) - (a + c)) + cx,$$

so that

$$\begin{aligned} e^\dagger &= \int_0^\infty H(x)l(x)dx = \\ &= \frac{1}{b}\left(\int_0^\infty \mu(x)l(x)dx - (a+c)\int_0^\infty l(x)dx\right) + c\int_0^\infty xl(x)dx = \\ &= \frac{1}{b}\left(1 - (a+c)e_0\right) + ce_0\bar{x}, \end{aligned}$$

from which (1) directly follows.
Q.E.D.

3. History and related results

Vaupel (1986) found that if the force of mortality follows a Gompertz curve, then $e^\dagger \approx 1/b$. This is a special case of (1), with $c = 0$ and ae_0 being small (as is the case for modern human populations). The generalization derived here is new.

Life table entropy $\bar{H}$ is the elasticity of life expectancy with respect to a proportional change in mortality. This was first derived by Leser (1955), and was restated by Keyfitz (1977a, 1977b) in continuous formulation. Demetrius (1974, 1975, 1976, 1979) applied information theory in both biology and demography, and $\bar{H}$ was one of the quantities he used. Hakkert (1987) and Hill (1993) compared $\bar{H}$ to other inequality measures. Mitra (1978), Goldman and Lord (1986), and Vaupel (1986) independently derived the mathematical expression for life disparity $e^\dagger = \int_0^\infty e(x)l(x)\mu(x)dx$, and showed that $\bar{H} = e^\dagger/e_0$. Vaupel and Canudas Romo (2003) showed that the derivative of life expectancy over time is given by the product of $e^\dagger$ and the rate of progress in reducing age-specific death rates.

Newer research on $e^\dagger$ includes Vaupel (2010) and Vaupel, Zhang, and van Raalte (2011). In the former paper, Vaupel showed that $e^\dagger - e_0$ is the total incremental change in remaining life expectancy over life, while in the latter paper the authors presented an international comparison of e_0 and $e^\dagger$ using life table data. In addition, several publications contain a perturbation analysis of $e^\dagger$; see Zhang and Vaupel (2009), Wagner (2010), and van Raalte and Caswell (2013).

4. Applications

Relationship (1) is a simple expression for life table entropy $\bar{H}$ of the Gompertz-Makeham model, and it relies solely on $\bar{\mu} = 1/e_0$, $\bar{x}$ and the parameters of the model.

The quantities included in (1) can be interpreted as follows:

$$\bar{\mu} - \mu_0$$

is the average absolute change in mortality and

$$b = \frac{d(\mu(x) - c)/dx}{\mu(x) - c}$$

is the rate of change in age-dependent mortality. Since

$$\bar{x} = \bar{e} = \frac{\int_0^\infty e(x) l(x) dx}{\int_0^\infty l(x) dx},$$

i.e., average age equals average remaining life expectancy in the stationary population (see Goldstein 2009),

$$c\bar{x} = \frac{\bar{e}}{c^{-1}}$$

is the ratio of the average remaining life expectancy in the stationary population to age-independent life expectancy (i.e., life expectancy as it would be if only the age-independent mortality component c remained). In the case of $c = 0$, it holds that $\bar{H}$ is just the ratio of average absolute ($= \bar{\mu} - \mu_0$) and relative ($= b$) increase in mortality. When the Gompertz-Makeham model applies (e.g., in cases of adult mortality in most countries), relationship (1) can be used to directly relate differences in the above quantities (and the parameters of the model) to differences in $\bar{H}$. The relationship could therefore help to shed new light on the factors that drive changes in lifespan inequality.

Relationship (1) can also be used to extend a result from Wrycza and Baudisch (2012). They gave the sensitivities of Gompertz-Makeham life expectancy with respect to a change in parameters a and b as

(2) $$\frac{de_0}{da} = \frac{1}{a}(c\bar{x}e_0 - e^\dagger) = \frac{e_0}{a}(c\bar{x} - \bar{H})$$

and

(3) $$\frac{de_0}{db} = \frac{1}{b}(e^\dagger - e_0) = \frac{e_0}{b}(\bar{H} - 1).$$

Plugging (1) into (2) and (3) gives

$$\frac{de_0}{da} = -\frac{e_0}{a}\frac{\bar{\mu}-\mu_0}{b} = \frac{1}{ab}\left(\frac{\mu_0}{\bar{\mu}} - 1\right) \tag{4}$$

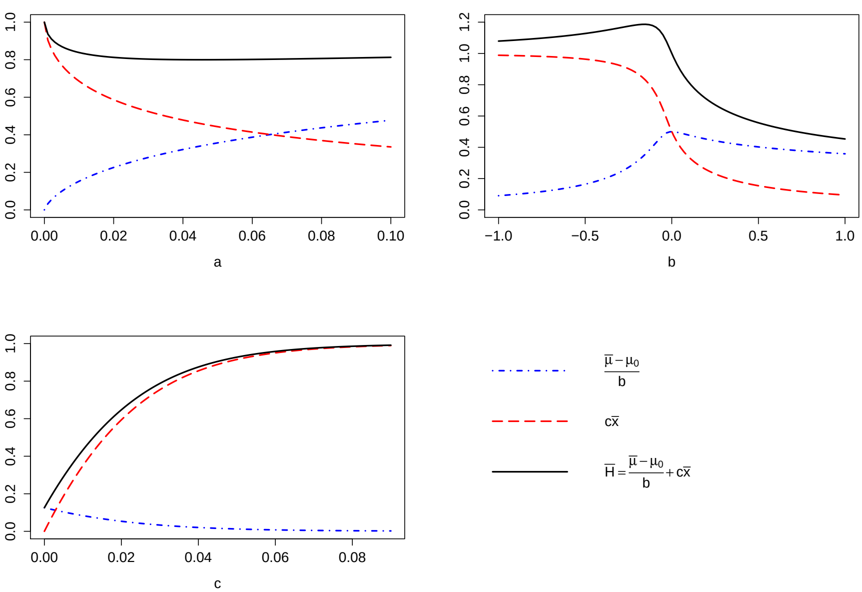
and

$$\frac{de_0}{db} = \frac{e_0}{b}\left(\frac{\bar{\mu}-\mu_0}{b} + c\bar{x} - 1\right) = \frac{1}{b^2}\left(1 - \frac{\mu_0}{\bar{\mu}} + be_0(c\bar{x}-1)\right). \tag{5}$$

These expressions are more explicit than the original ones.

To illustrate (1), I assume Gompertz-Makeham mortality and investigate what happens if two of the parameters are held fixed, while the remaining one takes different values. Figure 1 gives the graph of $\frac{\bar{\mu}-\mu_0}{b}$, $c\bar{x}$ and their sum $\bar{H}$, with the varying parameter being a (upper left plot), b (upper right plot), and c (lower left plot), respectively.

Figure 1: **Upper left:** $b = c = 0.1$**,** $0 \leq a \leq 0.1$**;**
Upper right: $a = c = 0.1$**,** $-1 \leq b \leq 1$**;**
Lower left: $a = 0.00002$**,** $b = 0.1$**,** $0 \leq c \leq 0.09$



We can see that for an increasing a and $b = c = 0.1$, the value of $\frac{\bar{\mu}-\mu_0}{b}$ is increasing while the value of $c\bar{x}$ is decreasing, resulting in a bathtub-shaped curve for $\bar{H}$.

For $a = c = 0.1$ and b increasing from -1 to 1, the curve of $c\bar{x}$ has a reverse sigmoid shape with $c\bar{x} \approx c \cdot \frac{1}{c} = 1$ for $b = -1$, $c\bar{x} = c/(a+c) = 0.5$ for $b = 0$, and $c\bar{x}$ decreasing for higher values of b. $\frac{\bar{\mu}-\mu_0}{b}$ follows a hump-shaped pattern with the maximum at $b = 0$. Adding up these two patterns results in a hump-shaped pattern for $\bar{H}$. However, the maximum here is reached *not* at $b = 0$, but at some negative value of b, meaning that entropy is highest for a specific pattern of negative senescence.

For $a = 0.00002$, $b = 0.1$ (i.e., for values similar to the values encountered in modern human populations) and c increasing from 0 to 0.09, $c\bar{x}$ is increasing from 0 to a value close to 1 in a concave pattern, while $\frac{\bar{\mu}-\mu_0}{b}$ is decreasing in a convex pattern from a value close to 0.2 to a value close to 0. Adding up these curves thus results in a concavely increasing pattern for $\bar{H}$, starting at a value close to 0.2 and approaching 1 for higher values of c.

Relationship (1) allows for a simple calculation of $e^\dagger$ and entropy $\bar{H}$ in the important case of Gompertz-Makeham mortality. It shows how these quantities relate to $e_0 = \frac{1}{\bar{\mu}}$ and $\bar{x}$. As the examples above illustrate, this could provide us with a more systematic understanding of the dynamics between the parameters a, b, and c on the one hand; and the demographically relevant quantities e_0, $\bar{x}$, $e^\dagger$, and $\bar{H}$ on the other.

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