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Research Article

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Abstract

BACKGROUND
Taylor's law (TL) states a linear relationship on logarithmic scales between the variance and the mean of a nonnegative quantity. TL has been observed in spatiotemporal contexts for the population density of hundreds of species including humans. TL also describes temporal variation in human mortality in developed countries, but no explanation has been proposed.

OBJECTIVE
To understand why and to what extent TL describes temporal variation in human mortality, we examine whether the mortality models of Gompertz, Makeham, and Siler are consistent with TL. We also examine how strongly TL differs between observed and modeled mortality, between women and men, and among countries.

METHOD
We analyze how well each mortality model explains TL fitted to observed occurrence–exposure death rates by comparing three features: the log–log linearity of the temporal variance as a function of the temporal mean, the age profile, and the slope of TL. We support some empirical findings from the Human Mortality Database with mathematical proofs.

RESULTS
TL describes modeled mortality better than observed mortality and describes Gompertz mortality best. The age profile of TL is closest between observed and Siler mortality. The slope of TL is closest between observed and Makeham mortality. The Gompertz

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model predicts TL with a slope of exactly 2 if the modal age at death increases linearly with time and the parameter that specifies the growth rate of mortality with age is constant in time. Observed mortality obeys TL with a slope generally less than 2. An explanation is that, when the parameters of the Gompertz model are estimated from observed mortality year by year, both the modal age at death and the growth rate of mortality with age change over time.

CONCLUSIONS
TL describes human mortality well in developed countries because their mortality schedules are approximated well by classical mortality models, which we have shown to obey TL.

CONTRIBUTION
We provide the first theoretical linkage between three classical demographic models of mortality and TL.

1. Introduction

Taylor's law (TL) states that the logarithm of the variance of some nonnegative quantity is approximately a linear function of the logarithm of the mean of that quantity in multiple sets of observations. TL describes the population densities of hundreds of species in ecology (Taylor 1961; Kilpatrick and Ives 2003; Kendal 2004) and many other nonnegative quantities (reviewed by Eisler, Bartos, and Kertész 2008), such as numbers of cancer metastases, numbers of cases of infectious diseases, numbers of single-nucleotide polymorphisms, sizes of tornado outbreaks (Tippett and Cohen 2016), and prime numbers (Cohen 2016). In human demography, TL describes the density (people per area) of Norway's population (Cohen, Xu, and Brunborg 2013) and the age-specific force of mortality (henceforth, simply 'mortality') in developed countries (Bohk, Rau, and Cohen 2015). A restrictive view of TL limits its application to apparently random variability without dominating systematic trends, that is, to what is sometimes called 'fluctuation scaling.' A broader view of TL, adopted here, includes applications of TL to nonnegative quantities that may change deterministically, for example, in purely mathematical structures (Kendal and Jørgensen 2011; Kendal 2013; Cohen 2013, Kendal and Jørgensen 2015; Xiao, Locey, and White 2015; Cohen 2016) or that may fluctuate apparently chaotically or randomly to some extent while dominated by systematic trends (Cohen, Xu, and Brunborg 2013; Bohk, Rau, and Cohen 2015). We discuss some unanswered questions raised by this broader view of TL in the concluding section 4.2 on future research.

As an empirical generalization, TL invites attempts at explanation. Why is there an approximately linear relationship between the log of the temporal mean and the log of the temporal variance of mortality in many developed countries? Is this empirical regularity in human mortality just a coincidence, or does it come from an underlying pattern or mechanism? Here we answer these questions by comparing the completely empirical results of Bohk, Rau, and Cohen (2015) with the results of fitting three human mortality models: those of Gompertz (1825), Makeham (1860), and Siler (1979, 1983), which belong to the same family. We show that fitted mortality of these models obeys TL, exactly or approximately. Hence these models provide a theoretical basis for TL in human mortality.

Mortality models express mathematically the age schedule of mortality (that is, mortality as a function of age) in a given year. Models differ in the number of parameters they use and in the age ranges for which they model mortality well. The more parameters they use, the more flexibly they can fit mortality at different ages, but the more difficult they are to analyze mathematically. Gompertz's model (1825), with two parameters, is one of the most popular models in demography. It assumes a linear increase in the logarithm of mortality with age. It usually describes well mortality at ages 30 to 90 (so-called senescent mortality). The model of Makeham (1860) adds to Gompertz's model an additional age-independent constant to represent nonsenescent background mortality effective at all ages. This improves the fit at some younger ages. The model of Siler (1979, 1983) adds to the Makeham model an exponential decay in mortality to represent the decline in mortality from infancy to childhood. We discuss still more complicated mortality models in the concluding section 4.2 on future research.

The main objectives of this study are to examine whether the models of Gompertz, Makeham, and Siler are consistent with TL and can help to explain why and to what extent TL holds. In addition, we examine how strongly TL differs between observed mortality and model mortality schedules, between women and men, and among countries.

We base our analysis on empirical occurrence–exposure death rates, statistical analysis, and theoretical explanations, and thus provide the first theoretical linkage between three classical demographic models of mortality and TL: We show mathematically that the Gompertz model (the simplest of the three models we considered) with linearly changing modal age at death and a constant rate of growth of mortality with age predicts TL with a slope that is exactly equal to 2. As the Gompertz model is a special case of the other two models, it is evident that the results from the Gompertz model are also valid for certain parameter values of the more complex models of Makeham and Siler. We find that observed mortality obeys TL with a slope generally less than 2, and that, when the parameters of the Gompertz model are estimated from observed mortality year by year, both the modal age at death and the growth rate of mortality with age change over time.

Our analyses yield theoretical and empirical insights into the occurrence of TL in human mortality, giving a comprehensive picture of the extent to which TL describes the temporal variance in age-specific mortality in human populations. As we have confirmed TL to be a regular pattern (rather than a coincidence) in human mortality, it can be validly applied in other demographic studies such as generating and evaluating mortality forecasts.

In the remainder of this article, Section 2 describes data and methods; Section 3 presents results; and Section 4 summarizes the main findings. Appendices 1 and 2 give theorems, mathematical proofs, and approximations of TL for the Gompertz and Makeham models respectively. Supplementary material (online) includes estimates of model parameters, figures, and descriptive files.

2. Data and methods

We extracted deaths and life-years of exposures to the risk of death by single year of age, 0 to 100, and calendar year, 1960 to 2009, for 12 countries of the Human Mortality Database (2015). Given this data, we estimated observed mortality (or 'observations') defined as deaths divided by exposures by single years of age and calculated predicted mortality for each year separately with the models of Gompertz, Makeham, and Siler. Henceforth the word 'observations' means 'occurrence–exposure death rates'.

2.1 Three theoretical models of mortality

We used mathematical expressions of the models of Gompertz, Makeham, and Siler that are based on the (old age) modal age at death (Horiuchi et al. 2013; Missov et al. 2015; Bergeron-Boucher, Ebeling, and Canudas-Romo 2015). The modal age at death is the age (beyond infancy and childhood) at which the probability density of life table deaths has a maximum. The conventional formulas for the models of Gompertz and Makeham use a parameter for an initial level of mortality instead of the modal age at death. The formulas based on the modal age at death, given below, are numerically more stable, have a better demographic interpretation, and are more comparable across populations and points of time (Horiuchi et al. 2013; Missov et al. 2015).

The Gompertz model expresses mortality μ at age x in year t as

$$\mu_{x,t} = \beta_t e^{\beta_t (x - M_t)}, \beta_t > 0, M_t > 0, \text{for } t = 1, \ldots, T, x = 0, \ldots, X. \quad (1)$$

Here β_t is the growth rate of mortality with age, M_t the modal age at death in year t, T the number of years of observations, and X the maximum observed age. The Gompertz model predicts, on a logarithmic scale of mortality and a linear scale of age, a linear increase in mortality from some young age to the oldest age $X = 100$ here.

The Makeham model expresses mortality μ at age x in year t as

$$\mu_{x,t} = c_t + \beta_t e^{\beta_t(x-M_t)}, c_t > 0. \tag{2}$$

The parameter β_t is the same as in the Gompertz model. Makeham added c_t to represent background mortality in year t, assumed to be the same at all ages x. The Makeham model predicts slowly increasing mortality from infancy through childhood to young adulthood; thereafter, it models a nearly linear increase in log mortality with increasing age.

The Siler model expresses mortality μ at age x in year t as

$$\mu_{x,t} = \alpha_t e^{-\beta_{1,t}x} + c_t + \beta_{2,t} e^{\beta_{2,t}(x-M_t)}, \alpha_t > 0, \beta_{1,t} > 0. \tag{3}$$

Here $\beta_{2,t} = \beta_t$ of the Makeham and Gompertz models. Siler adds two additional parameters: α_t is infant mortality, and $\beta_{1,t}$ is the rate of decline with increasing age x of childhood mortality in year t. The Siler model predicts decreasing mortality from infancy to childhood, slowly increasing mortality from childhood to young adulthood, and nearly linearly increasing log mortality with increasing age throughout adulthood.

2.2 Taylor's law, mean and variance of mortality

A temporal TL describes a linear relationship of $\log_{10}$ of temporal variance (variance over time) to $\log_{10}$ of temporal mean (mean over time) of mortality μ at age x:

$$log_{10}Var(\mu_x) = a + b \cdot log_{10}E(\mu_x), \tag{4}$$

where a is the intercept and b is the slope. With T years of observations or theoretical (fitted) values of mortality, the temporal mean of mortality μ at age x is

$$E(\mu_x) = \frac{1}{T}\sum_{t=1}^{T} \mu_{x,t}, \tag{5}$$

and the temporal variance is

$$Var(\mu_x) = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_{x,t} - E(\mu_x)\right)^2. \tag{6}$$

We plot (on log–log coordinates) temporal variances and means of mortality for each age x in one national population at a time, separately for different national populations. These plots depict so-called cross-age-scenarios of TL (Bohk, Rau, and Cohen 2015).

2.3 Statistical methods and visualization

2.3.1 Parameter estimation

We estimated the values of the parameters of the models of Gompertz, Makeham, and Siler from deaths and exposures using the method of maximum likelihood. Specifically, we maximized a Poisson log-likelihood in R (2015) with the function DEoptim (Mullen et al. 2011).

2.3.2 Analysis of log–log linearity

To analyze how closely TL approximates the log temporal mean and log temporal variance of observed and/or modeled mortality, we used the linear correlation coefficient r^2. The closer r^2 is to one, the better TL describes the relation of log temporal variance to log temporal mean of mortality.

2.3.3 Visualizing the age profile of TL

To compare the age profiles between observed and modeled mortality, we plotted the $\log_{10}$ temporal variance of mortality as a function of the $\log_{10}$ temporal mean by single-year age groups using yellow and orange to represent children (0 to 20 years of age), red and magenta for adults (21 to 60 years of age), and blue and green for older ages (61 to 100 years of age).

2.3.4 Analysis of covariance

To analyze how well a theoretical model predicts the slope of TL fitted to observed mortality, we used the analysis of covariance.

1. To determine whether the slopes of TL differ between observed and modeled mortality (by sex, country, and model), we estimated a linear regression with an interaction term between the variables $log_{10}E(\mu_x)$ and a dichotomous variable *model* (with values 'observed' and 'model') using the *lm()* function in R:

$$log_{10}Var(\mu_x) = c_0 + c_1 log_{10}E(\mu_x) + c_2 model + c_3(log_{10}E(\mu_x) \times model). \quad (7)$$

We use observed mortality as the reference level for the categorical variable *model*. The null hypothesis was that $c_3 = 0$, that is, that there was no difference between the model and the observations in the slope of $log_{10}Var(\mu_x)$ as a linear function of $log_{10}E(\mu_x)$. A very low p-value indicated that the coefficient c_3 of the interaction term is non-zero, and that the slope of TL of a model life table is not equal to the slope of TL of observed mortality data.

2a. We also used analysis of covariance to determine whether the slopes of TL differ between males and females (by country and mortality model). To analyze if the slopes of TL differed between males and females, we estimated a linear regression with an interaction term between the variables $log_{10}E(\mu_x)$ and a dichotomous variable *sex* (with values 'male' and 'female'):

$$log_{10}Var(\mu_x) = d_0 + d_1 log_{10}E(\mu_x) + d_2 sex + d_3(log_{10}E(\mu_x) \times sex). \quad (8)$$

We used female mortality as the reference level. For the null hypothesis that $d_3 = 0$, a very low p-value indicated that the coefficient d_3 of the interaction term is non-zero, and that the slope of TL is different between males and females.

2b. To analyze if sex differences in the slope of TL differ between observed and modeled mortality (by country and model), we estimated a linear regression with pairwise and three-way interaction terms among the variables $log_{10}E(\mu_x)$, *sex,* and *model.* Here, unlike equation (7) above, the variable *model* had four values: 0 (observed), 1 (Gompertz), 2 (Makeham), 3 (Siler):

$$\begin{aligned} log_{10}Var(\mu_x) = {} & f_0 + f_1 log_{10}E(\mu_x) + f_2 sex + f_3 model \\ & + f_4(log_{10}E(\mu_x) \times sex) + f_5(log_{10}E(\mu_x) \times model) + f_6(sex \times model) \\ & + f_7(log_{10}E(\mu_x) \times sex \times model). \end{aligned} \quad (9)$$

Observed mortality was the reference level for model, as in equation (7), and female mortality was the reference level for sex. The null hypothesis was that $f_7 = 0$, that is, that model had no influence on the interaction between *sex* and $log_{10}E(\mu_x)$.

2.4 Availability of data and supplementary information

In addition to the data on mortality, which is publicly available from the Human Mortality Database (2015), supplementary information deposited with this paper includes a spreadsheet (TLinMortalityModels.csv) with the values of 41 variables and a text file (TLinMortalityModels-Documentation-Variables.txt) that defines these variables. The spreadsheet gives the parameter estimates of the models of Gompertz, Makeham, and Siler for observed female and male mortality from 1960 to 2009 in 12 countries of the Human Mortality Database (2015). This information is graphed in Figures 1–17 and supplementary figures A-1–A-20. R code is available at: https://github.com/christina-bohk-ewald/taylor-law-mortality.

3. Results

3.1 Fitted mortality of Gompertz, Makeham, and Siler models

Before we analyze how consistent the models of Gompertz, Makeham, and Siler are with TL, we first show how well they fit observed mortality. Figures 1 and 2 display the observed age-specific mortality (on a logarithmic scale) as a function of age from 0 to 100, for women and men, respectively, from 1960 (light gray) to 2009 (black) in 12 countries of the Human Mortality Database (2015). Fitted mortality is depicted in green, blue, and red for the models of Gompertz, Makeham, and Siler, respectively, in 2009 for each country.

The observed mortality shows a typical age pattern: a fall from infancy to around age 10, an 'accident bump' around age 20 (often more pronounced for men than for women), and a roughly linear rise on the log scale beyond age 30. Mortality at each age declined with time in many developed countries since 1960. The declines occurred mainly at younger ages before they spread towards higher ages (Christensen et al. 2009; Rau et al. 2008; Vaupel 2010).

Figure 1: Observed female mortality from 1960 (light gray) to 2009 (black), and fitted female mortality with models of Gompertz (green), Makeham (blue), and Siler (red) in 2009 for 12 countries of the Human Mortality Database (2015)

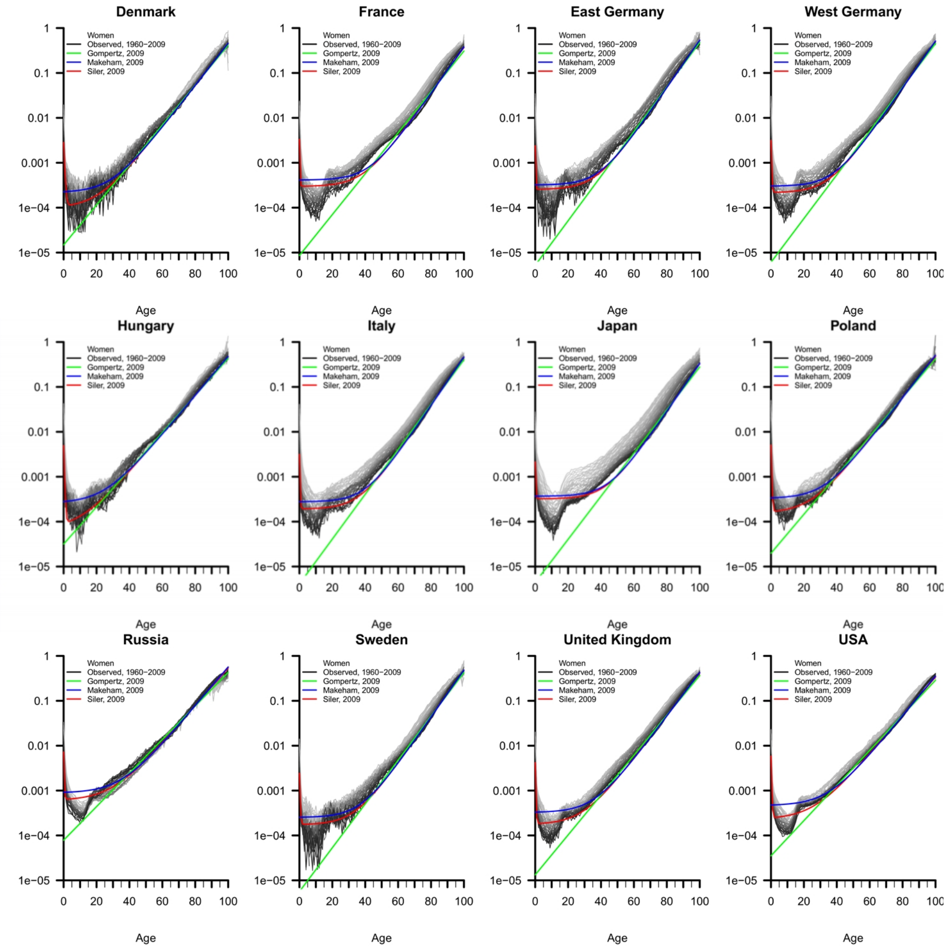
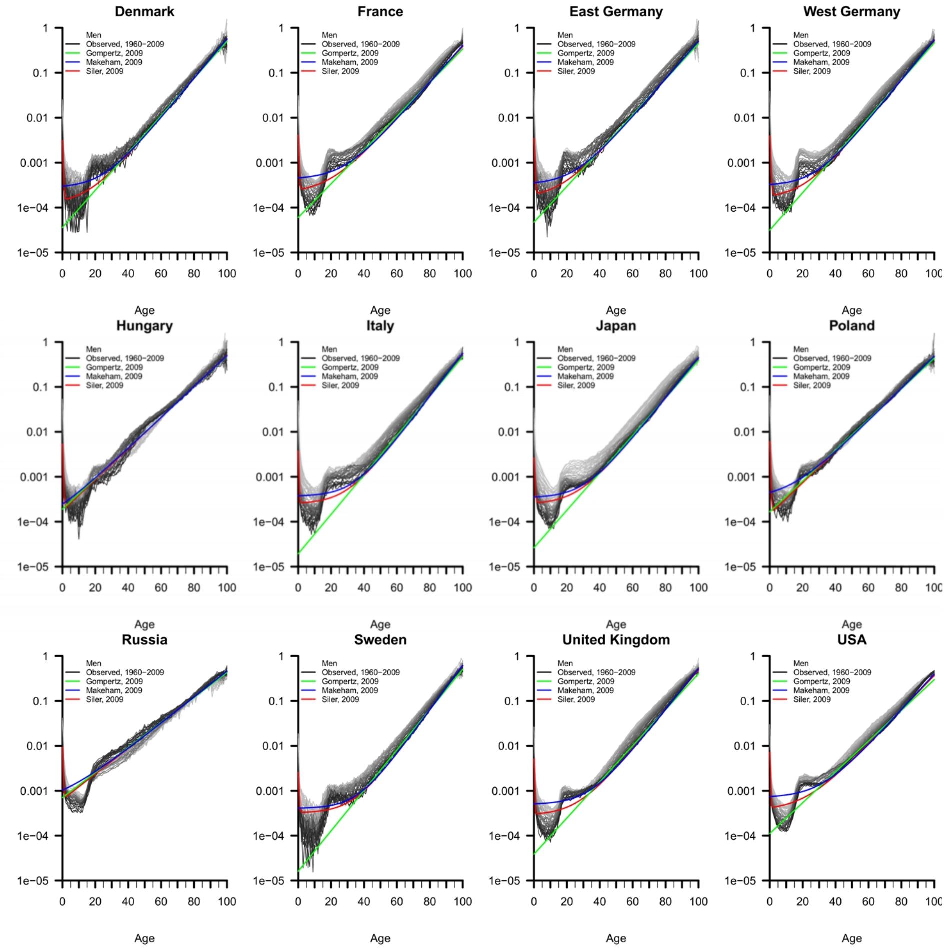


Figure 2: Observed male mortality from 1960 (light gray) to 2009 (black), and fitted male mortality with models of Gompertz (green), Makeham (blue), and Siler (red) in 2009 for 12 countries of the Human Mortality Database (2015)



The Gompertz model fits observed mortality better at adult and older ages than at younger ages, where the predicted mortality is systematically and substantially too low. The Makeham model fits better than the Gompertz model, particularly for younger ages, but the modeled mortality increases monotonically with age, unlike the observations. The Siler model fits the age profile of mortality better than the Gompertz and Makeham models. None of the models reproduces the observed ‘accident bump’ of excess mortality of young adult ages. The findings in this paragraph confirm prior observations by others about the fit between human mortality data and the Gompertz, Makeham, and Siler models (e.g., Bongaarts 2005; Canudas-Romo 2008; Horiuchi et al. 2013; Bergeron-Boucher, Ebeling, and Canudas-Romo 2015).

3.2 Statistical and visual tests of TL in observed and fitted mortality

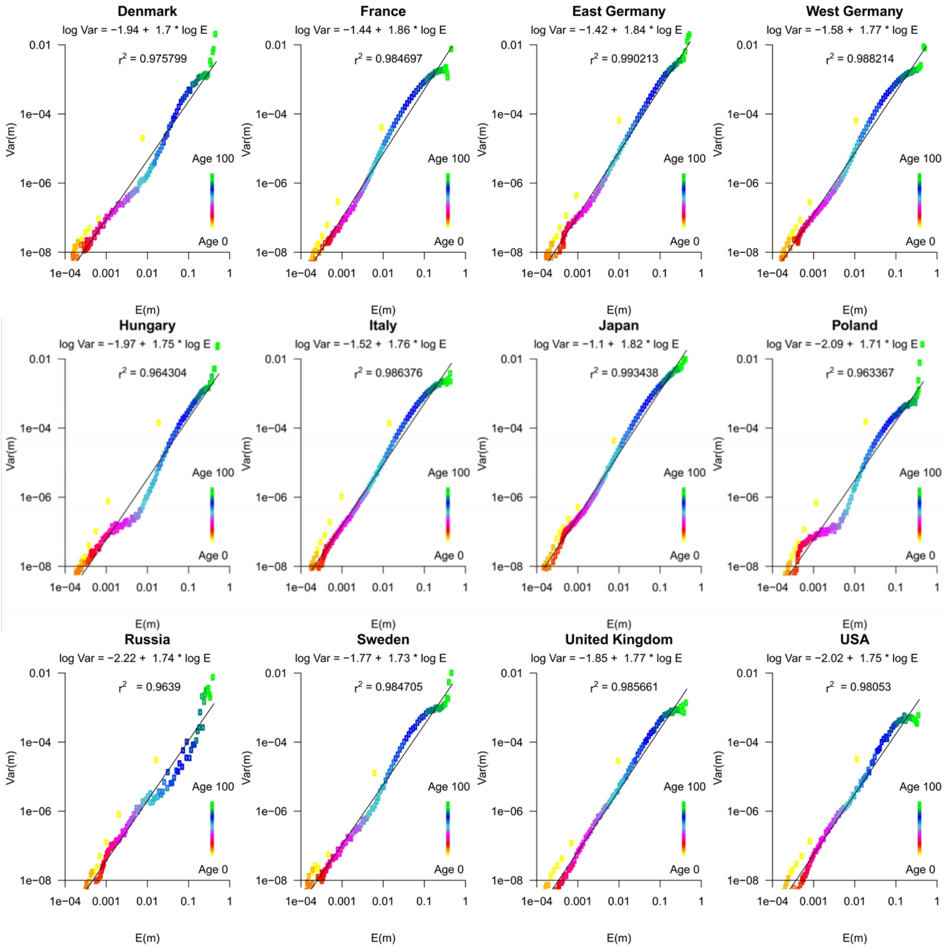
This section reports the statistical analysis and visual tests proposed in Section 2. Figures 3–10 display the cross-age-scenarios of TL for observed and fitted mortality of women and men, respectively, in 12 countries of the Human Mortality Database (2015). Odd-numbered figures are for females, even-numbered for males.

3.2.1 Log-log linearity and r^2 values

Figures 3–4 compare the temporal means and temporal variances of observed mortality with TL (the fitted least squares straight line), on log–log coordinates. In these figures, r^2 measures the linearity of observed log temporal variance as a function of observed log temporal mean. In Figures 5–10, r^2 measures the linearity of the log temporal variance of modeled mortality as a function of the log temporal mean of modeled mortality.

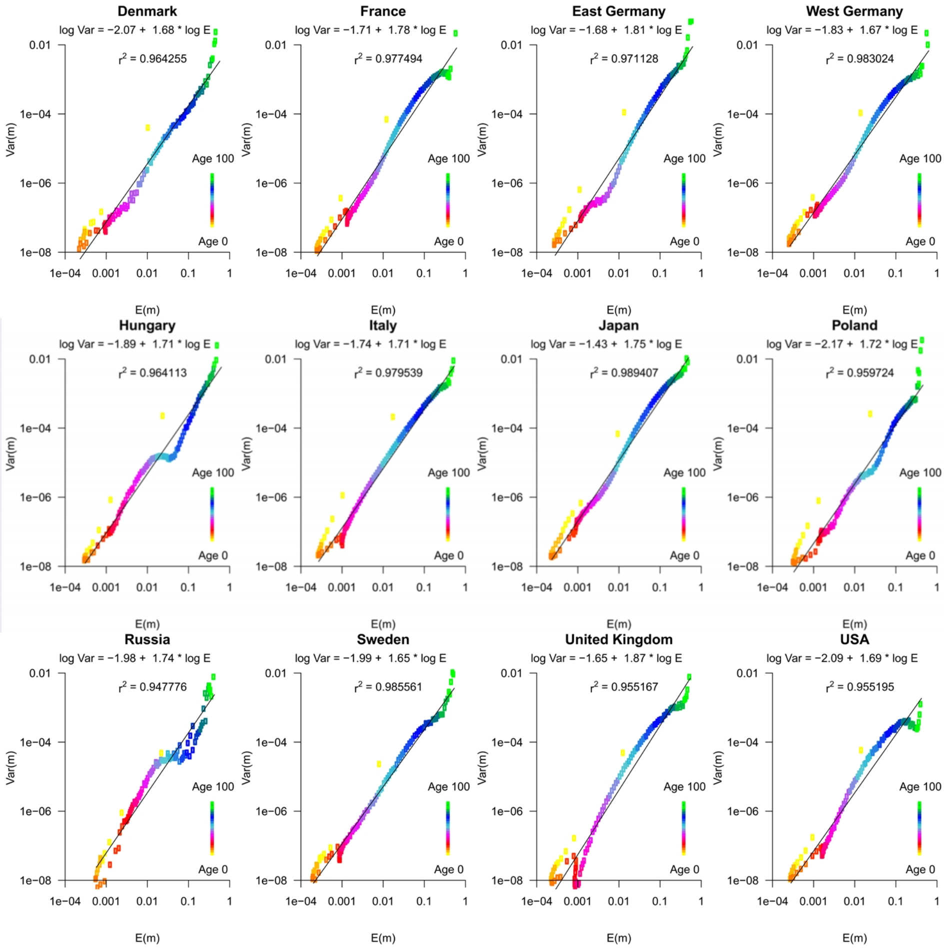
TL describes well the observed mortality (Figures 3, 4) and the mortality of the models of Gompertz (Figures 5, 6), Makeham (Figures 7, 8), and Siler (Figures 9, 10). For observations of women, $0.96 \leq r^2 \leq 0.99$, and of men, $0.95 \leq r^2 \leq 0.99$, in the selected countries. Where the mortality models had $r^2 \geq 0.999$, sometimes there was excellent agreement with the strictly linear relationship posited by TL (e.g., the Gompertz model for East Germany in Figure 5 and the Makeham model for France and Japan in Figure 7). In some cases the models, and particularly the Gompertz model, predicted a relation of log variance to log mean that was closer to linear than was the relation of log variance to log mean based on observed mortality. The Gompertz model had $r^2 \geq 0.99$ more often than the other two models.

Figure 3: TL (solid black line) in observed female mortality for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)



Note: This figure essentially reproduces Figure 2 of Bohk, Rau, and Cohen (2015), with the addition here of a color scale and greater precision in the values of r^2.

Figure 4: TL (solid black line) in observed male mortality for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)



Note: This figure essentially reproduces Figure A-1 of Bohk, Rau, and Cohen (2015), with the addition here of a color scale and greater precision in the values of r^2.

Figure 5: TL (solid black line) in fitted female mortality of the Gompertz model for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

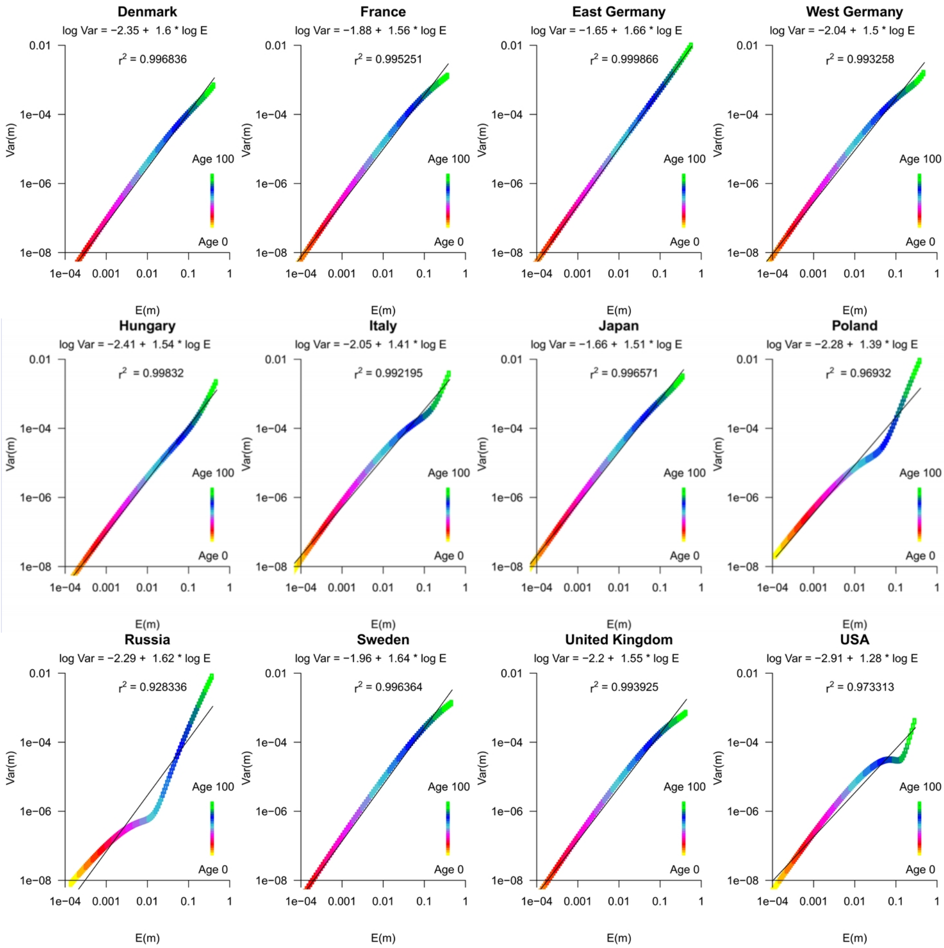


Figure 6: TL (solid black line) in fitted male mortality of the Gompertz model for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

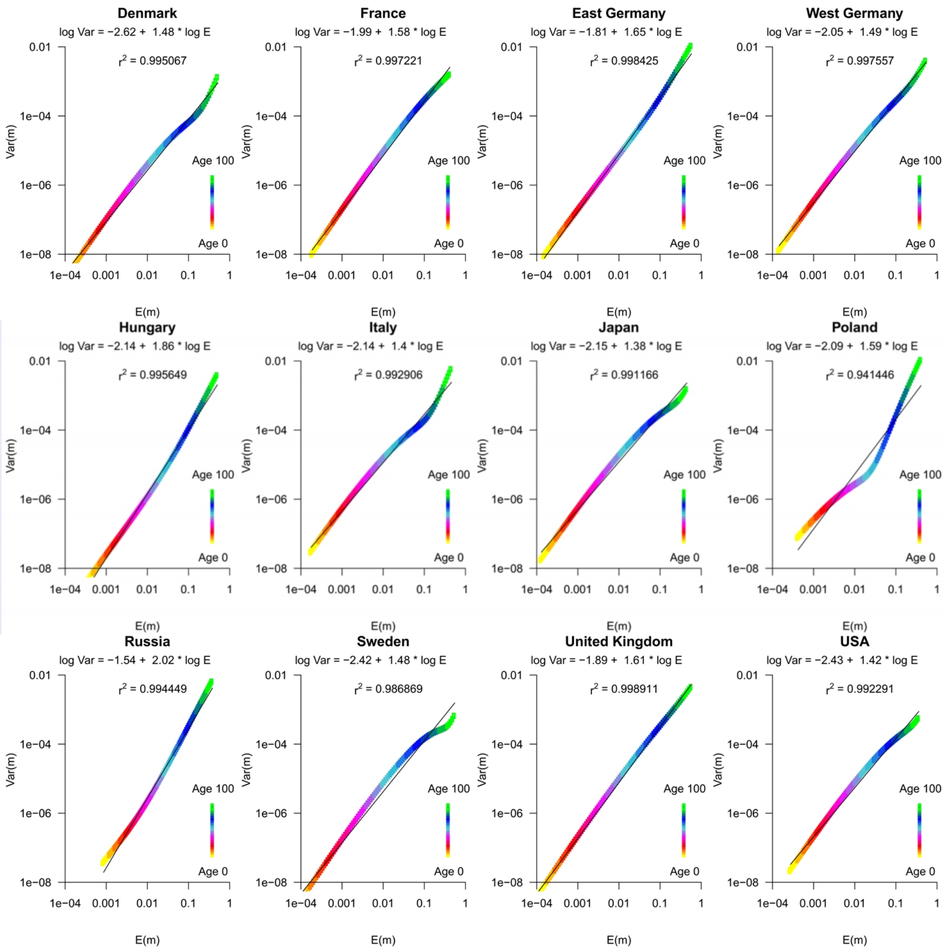


Figure 7: TL (solid black line) in fitted female mortality of the model of Makeham for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

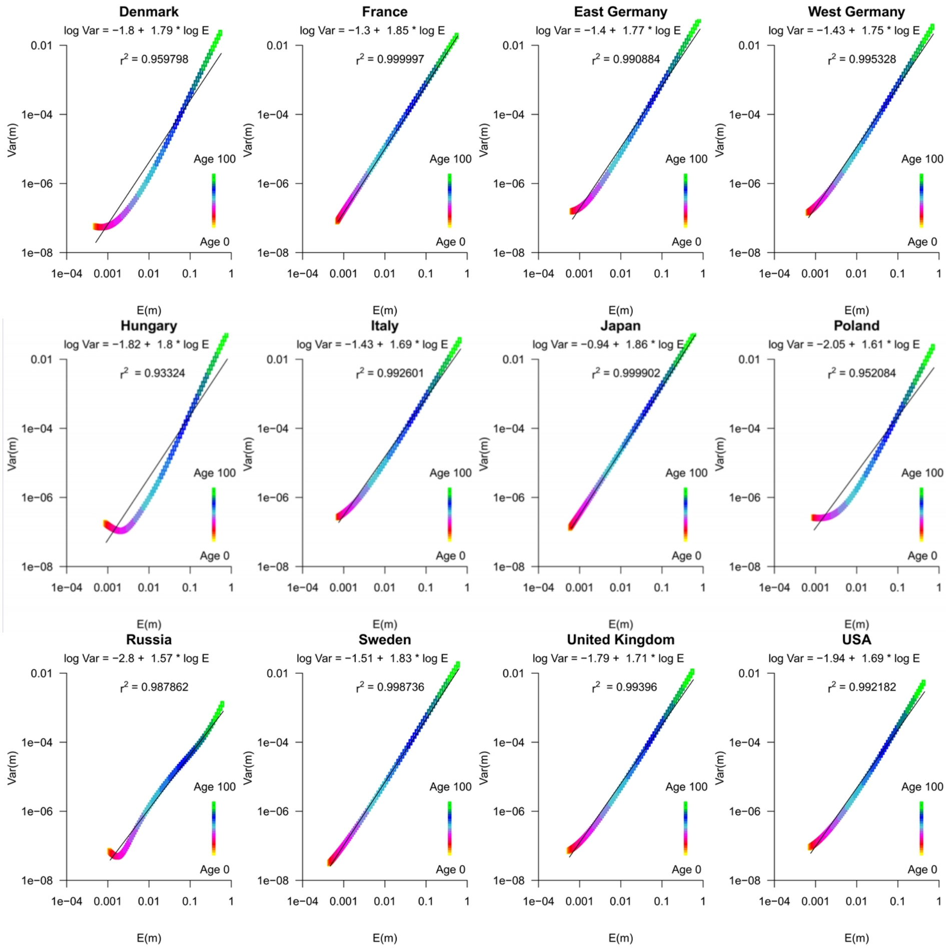


Figure 8: TL (solid black line) in fitted male mortality of the model of Makeham for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

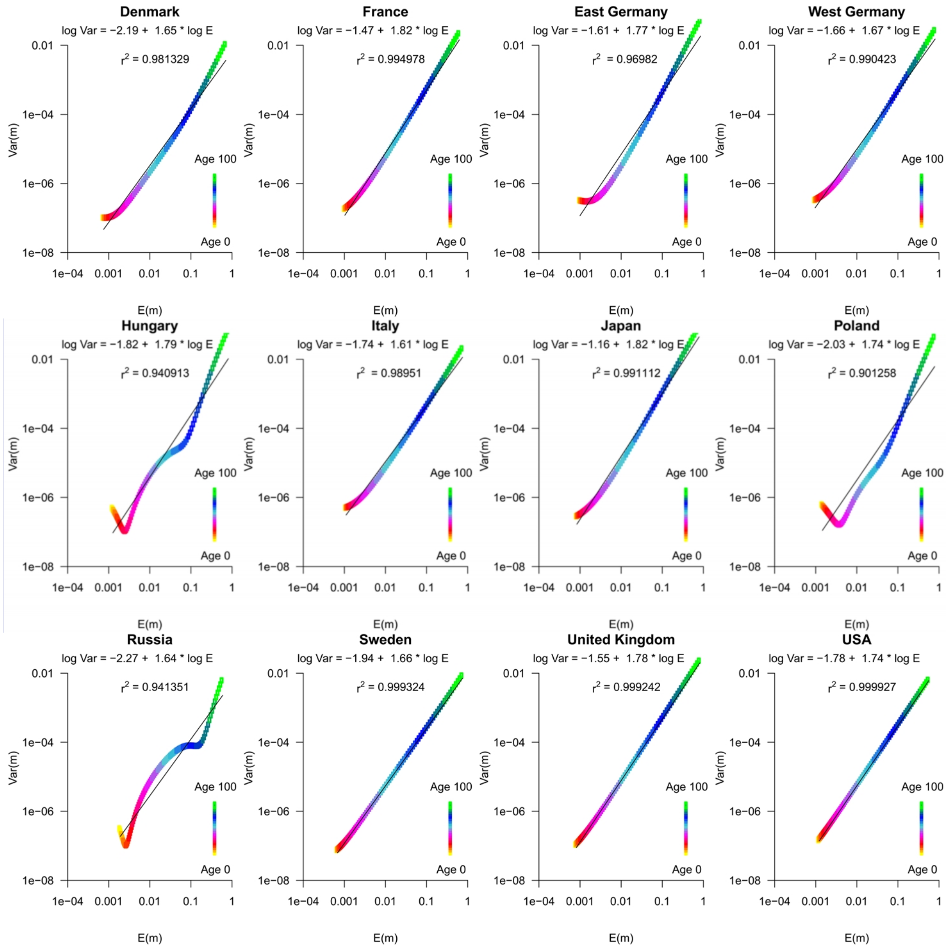


Figure 9: TL (solid black line) in fitted female mortality of the model of Siler for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

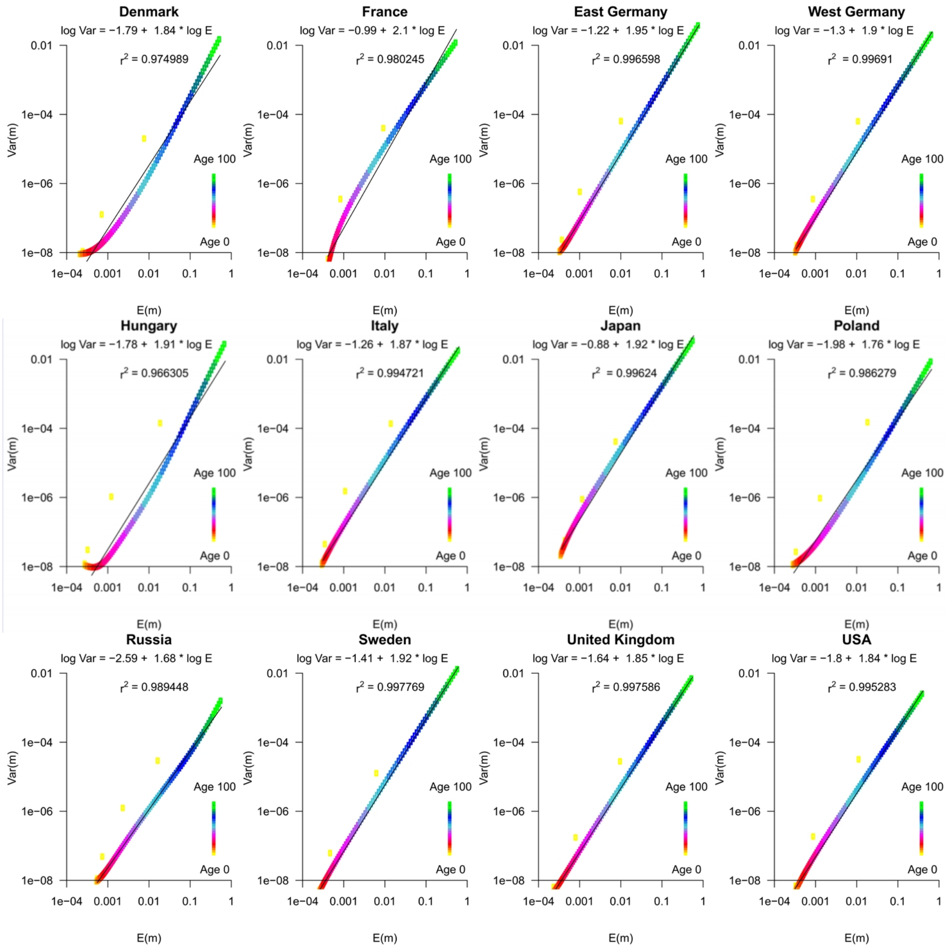
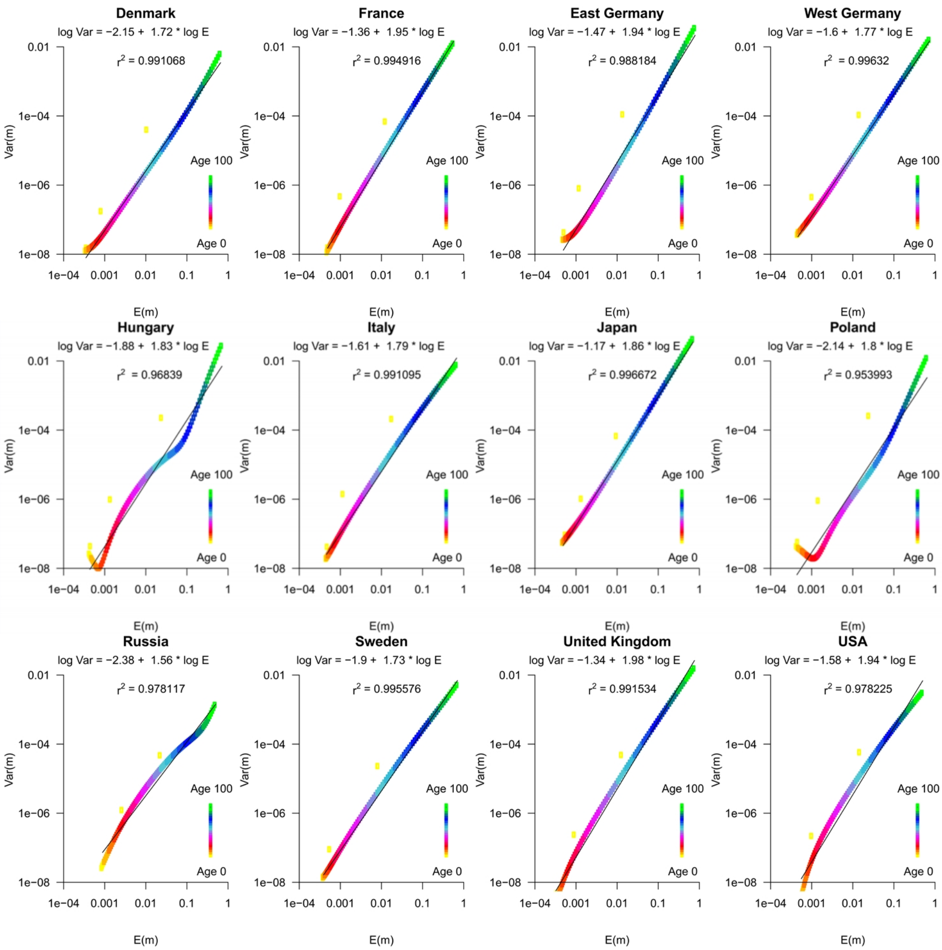


Figure 10: TL (solid black line) in fitted male mortality of the model of Siler for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)



According to the r^2 values, TL describes Gompertz mortality best among these three models. Appendix 1 gives a mathematical proof that TL describes Gompertz mortality exactly if the modal age at death M_t increases linearly in time and if the rate of growth of mortality with age β_t is constant in time. The first assumption is close to reality, as we shall see. The second assumption is not: β_t increased slightly over time within a narrow range, even though β_t is hypothesized to be constant across individuals and over time (Vaupel 2010). Nevertheless, the excellent agreement between TL and Gompertz mortality is at least partially explained by this mathematical analysis.

3.2.2 Visually comparing age profiles between observed and modeled mortality

In this section, we visually compare the age pattern of TL between observed and modeled mortality.

TL in observed mortality data (Figures 3–4) has a typical pattern that is similar for women and men in many populations. Both the $\log_{10}$ temporal variance and the $\log_{10}$ temporal mean of mortality increase linearly from young adulthood (red) to the elderly (green), and they decrease from infancy (yellow) to childhood (orange). The changes in the $\log_{10}$ temporal mean are expected from the increasing mortality with age at older ages and the decreasing mortality with age from infancy to childhood. The corresponding linear changes in the $\log_{10}$ temporal variance were not known prior to the work of Bohk, Rau, and Cohen (2015).

TL of Gompertz mortality (Figures 5–6) mirrors the pattern of TL of the observations well at adult and old ages. However, both the $\log_{10}$ temporal variance and the $\log_{10}$ temporal mean of mortality of infants and children (yellow to orange) are modeled to be smaller than those of young adults (red), unlike the observations. This major difference between observed mortality and the Gompertz model arises because the Gompertz model assumes a log-linear increase in mortality from the youngest to the oldest age. The Gompertz model captures neither declining mortality from infancy to childhood nor its related effect on TL.

TL of Makeham mortality (Figures 7–8) mirrors the pattern of TL of the observations well at adult and old ages. However, both the $\log_{10}$ temporal variance and the $\log_{10}$ temporal mean of mortality are modeled to be almost equal for infants and children (yellow to orange) on the one side, and young adults (red) on the other side, unlike the observed mortality and Gompertz mortality. These major differences arise because the Makeham model assumes that mortality increases slowly from infancy to young adulthood and increases exponentially thereafter. As a consequence, the Makeham model captures neither the decline in observed mortality from infancy to childhood nor its related effect on TL. As expected, TL of Makeham mortality is closer than TL of Gompertz mortality to TL of observations.

TL of Siler mortality (Figures 9–10) mirrors reasonably well the pattern of TL of the observations for all ages in most of the 12 countries. The term in the Siler model that models declining mortality from infancy to childhood captures the related effect on TL.

From visually comparing the age profiles, we conclude that the TL of the Siler model (fitted to observed mortality) is closest to the TL of observed mortality.

3.2.3 Slopes of TL

In this section, we compare the slopes of TL between observed and modeled mortality.

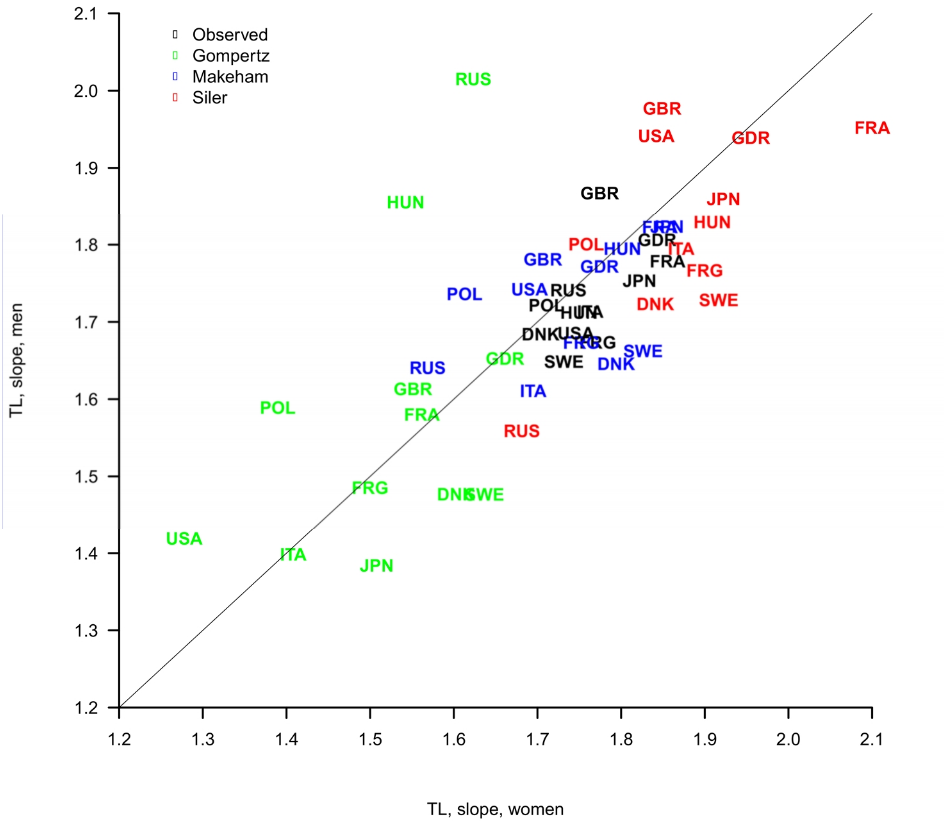
A. Visual overview

Figure 11 displays the slope of TL for women on the horizontal axis and the slope of TL for men on the vertical axis for 12 countries of the Human Mortality Database (2015). Slopes are estimated from observed mortality (black) and from the fitted models of Gompertz (green), Makeham (blue), and Siler (red). Figure 11 summarizes 96 slopes (96 = 12 × 4 × 2). Only two slopes exceed 2 (for TL fitted to the Gompertz model for Russian males, $b = 2.02$; and for TL fitted to the Siler model for French females, $b = 2.1$). We regard these two slopes as outliers. All slopes exceed 1.

Slopes of TL estimated from observed mortality are closer to slopes estimated from the Makeham model than they are to the slopes estimated from the other two models. Women and men have greater slopes according to the Siler model than are estimated from observed mortality. Hence the Siler model assumes more rapid increases in the variance of mortality with increasing mean mortality than is observed. Women and men have substantially smaller slopes according to the Gompertz model than are estimated from observed mortality. The Gompertz model assumes slower increases in the variance of mortality with increasing mean mortality than is observed.

The diagonal line in Figure 11 represents equal TL slopes for women and men. Sex differences in slopes of TL are given in Figure 11 by vertical deviations above or below the diagonal. Sex differences appear to be slightly smaller for observed mortality and the Makeham model than for the Gompertz and Siler models. Exceptional outliers are the slopes of TL of Gompertz mortality for countries with relatively high mortality such as Russia, Hungary, and Poland (Grigoriev, Doblhammer-Reiter, and Shkolnikov 2013; Grigoriev et al. 2010; Shkolnikov et al. 2013).

Figure 11: Scatterplot of slope of TL for women (horizontal axis) and men (vertical axis) for 12 countries of the Human Mortality Database (2015)



Note: Observations are in black, fitted data of the models of Gompertz, Makeham, and Siler are depicted in green, blue, and red respectively.

B. Covariance analysis

We examine differences in the slopes of TL between sexes and models using covariance analysis.

B1. Does the slope of TL for observed mortality differ from the slopes of TL for models?

The p-values of the test for the significance of the interaction term c_3 in the analysis of covariance, eq. (7), are given by sex, country, and model (Gompertz, Makeham, Siler) in Table 1. This analysis confirms the findings from Figure 11. Specifically, the slope of TL of the Makeham model is not significantly different from the slope of TL of observed mortality for women and men in almost all of the 12 countries. By contrast, the slope of TL of the Siler model is significantly different from the slope of TL of observed mortality for women and men in almost all 12 countries. An exception is, for example, Poland. The slope of TL of the Gompertz model is significantly different from the slope of TL of observed mortality for women and men in almost all 12 countries.

Table 1: P-values to test the null hypothesis of no differences in TL slope between observed data and the fitted models of Gompertz, Makeham, and Siler, for women and men in 12 countries of the Human Mortality Database (2015)

Equal to TL of observed data?	Gompertz			Makeham			Siler		
	Women	Men	Both	Women	Men	Both	Women	Men	Both
All countries	0	0	0	0.073200	0.284200	0.032000	0	0	0
Denmark	0.005358	0	0	0.022461	0.231000	0.284805	0.000312	0.208000	0.000392
France	0	0	0	0.748900	0.076500	0.486783	0	0	0
East Germany	0	0.000006	0	0.000580	0.329880	0.011300	0	0.000120	0
West Germany	0	0	0	0.351730	0.986726	0.432000	0	0.000014	0
Hungary	0.000007	0.001720	0.005730	0.296356	0.078070	0.106000	0.000661	0.008880	0.000077
Italy	0	0	0	0.003660	0.000132	0.000004	0.000001	0.001448	0
Japan	0	0	0	0.035169	0.001420	0.001870	0	0	0
Poland	0	0.032000	0	0.021900	0.811000	0.141000	0.233800	0.177000	0.153
Russia	0.009401	0	0.789410	0.000337	0.028860	0.002170	0.213779	0.000048	0.01896
Sweden	0	0	0	0.000002	0.532277	0.001853	0	0.000214	0
United Kingdom	0	0	0	0.002140	0.009960	0.000722	0.000380	0.000650	0.000003
United States	0	0	0	0.051616	0.114000	0.970258	0.000395	0	0

Note: A p-value below 0.001 indicates that the coefficient of the interaction term is statistically significantly non-zero. An entry of 0 means that the rounded value of p is 0.000000.

B2a. Does the slope of TL differ between males and females for observed mortality and for models?

The p-values of the test for the significance of the interaction term d_3 in eq. (8) are given by country for observed mortality and models in Table 2.

Table 2: P-values to test the null hypothesis of no differences between females and males in the slope of TL fitted to observed death rates and in the slope of TL fitted to the models of Gompertz, Makeham, and Siler, in 12 countries of the Human Mortality Database (2015)

Sex differences in TL?	Observed data	Gompertz	Makeham	Siler
All countries	0.023190	0.087000	0.199000	0.000290
Denmark	0.644000	0	0.000844	0.000736
France	0.031800	0.204590	0.069400	0.000017
East Germany	0.306800	0.187000	0.943370	0.510000
West Germany	0.000968	0.336000	0.000136	0
Hungary	0.426000	0	0.920000	0.106000
Italy	0.126570	0.582600	0.000200	0.000514
Japan	0.003760	0	0.068200	0.000128
Poland	0.827000	0.000021	0.068100	0.335000
Russia	0.936100	0	0.095200	0.000035
Sweden	0.005600	0	0	0
United Kingdom	0.042600	0.000009	0	0
United States	0.171000	0	0.001040	0.001610

Note: A p-value below 0.001 indicates that the coefficient of the interaction term is statistically significantly non-zero. An entry of 0 means that the rounded value of p is 0.000000.

Slopes of TL fitted to observed mortality are not significantly different between males and females in almost all 12 countries. With $p = 0.001$, West Germany is the only exception. However, the slopes of TL differ between males and females almost as strongly in countries like France, the United Kingdom, Japan, and Sweden. The vertical deviations from the diagonal in Figure 11 are similar for these four countries.

The slopes of TL of modeled mortality differ significantly between males and females for many countries. These sex differences are slightly more pronounced in the models of Gompertz and Siler than in the Makeham model. This supports the finding from Figure 11.

East Germany is the only country for which the slopes of TL of observed and modeled mortality do not differ between males and females. All the points for this country are almost on the diagonal in Figure 11. We do not know if this agreement of male and female slopes indicates a problem in the mortality data or a statistical fluctuation or some special feature of East Germany.

B2b. Do the sex differences in the slope of TL differ between observed mortality and models?

Table 3 lists the p-values of the test for the significance of the coefficient f_7 of the three-way interaction among $log_{10}E(\mu_x)$, *sex,* and *model* of eq. (9). Consistent with the findings from Figure 11 and Table 2, the sex differences in the slope of TL do not differ much between the observed mortality and the models of Gompertz, Makeham, and Siler. Specifically, the sex differences in the slope of TL are not significantly different between observed mortality and the Makeham model in each of the 12 countries. The sex differences in the slope of TL differ significantly between observed mortality and the Gompertz model in only three countries: Hungary, Russia, and the United States; and, though less significantly, in Sweden, Japan, West Germany, France, and Denmark. The sex differences in the slope of TL differ significantly between observed mortality and the Siler model in only two countries: Sweden and the United States; and, though less significantly, in Russia, France, and Denmark.

Table 3: P-values to test the null hypothesis of no differences in the sex differences in TL slope between observed mortality rates and fitted models of Gompertz, Makeham, and Siler, in 12 countries of the Human Mortality Database (2015)

Sex differences in TL of models equal to sex differences in observed data?	Gompertz	Makeham	Siler
All countries	0.005060	0.581010	0.349870
Denmark	0.025680	0.011860	0.045340
France	0.008070	0.164250	0.056650
East Germany	0.447548	0.392768	0.586554
West Germany	0.003030	0.527880	0.237640
Hungary	0	0.648766	0.515661
Italy	0.226781	0.329177	0.414148
Japan	0.029082	0.165447	0.817667
Poland	0.00917	0.128900	0.650510
Russia	0	0.303197	0.046573
Sweden	0.004344	0.006243	0.000250
United Kingdom	0.401217	0.643469	0.354857
United States	0.000003	0.013959	0.000247

Note: A p-value below 0.001 indicates that the coefficient of the interaction term is non-zero. An entry of 0 means that the rounded value of p is 0.000000.

That the sex differences in the slope of TL are significant for Hungary, Russia, and Japan could be explained by the increasing sex gap in life expectancy at birth in those countries in the 1980s and 1990s. By contrast, other European countries experienced a decline in the female–male differences in mortality (Oksuzyan et al. 2008).

3.3 Mathematical proof and theoretical explanations for TL in Gompertz mortality

In this section, we prove mathematically that the Gompertz model can explain the form of TL under certain conditions. We investigate theoretically whether the Gompertz model can explain the observed parameter values of TL.

3.3.1 Gompertz mortality predicts TL with slope 2 under certain conditions

We prove mathematically (in Appendix 1) that the Gompertz model eq. (1) with modal age at death M_t increasing linearly in time and $\beta_t = \beta > 0$ obeys a cross-age-scenario of TL exactly with slope b = 2. Appendix 1 gives an explicit form for the intercept of TL and a detailed proof. This theorem gives analytically the exact relation between the parameters of the Gompertz model and the parameters of TL in one simple case.

3.3.2 Temporal trend of β_t alters the slope of TL fitted to Gompertz mortality

The theorem's assumptions that β_t is constant over time and that the modal age at death M_t increases linearly with time t cannot describe the reality of many countries since, empirically, the slope b of TL fitted to the temporal mean and the temporal variance of observed mortality was always less than 2 (Figures 3–4), ranging from 1.65 to 1.87.

The numerical estimates of the parameters of all three mortality models, separately for females and males, for all countries and years, along with the parameters of linear regressions of these parameters as functions of time, are given in the Supplementary spreadsheet and graphed in Supplementary Figures A-1–A-20.

In the Gompertz model, even if the mode M_t increases approximately linearly with time (as shown in Figures A-3–A-4), the coefficient β_t must change in time so that, with increasing age, the variance of mortality does not increase as fast as the square of the mean mortality. Empirically, the annual estimates of β_t increase approximately linearly over time for females (Figure A-1) and males (Figure A-2) for all 12 countries.

3.3.2.1 Theoretical analysis

We now analyze the impact of a temporal trend in β_t on the estimated slope b of a cross-age-scenario of TL fitted to Gompertz mortality rates. In the Gompertz model eq. (1), β_t appears twice, as a linear coefficient and in the exponent. We introduce separate notation for these two appearances of β_t so that we may analyze separately the two different effects of the temporal trend in β_t:

$$\mu_{x,t} = \beta_{t,down} e^{\beta_{t,up}(x-M_t)}.$$

We examine two cases:

Case 1: If $\beta_{t,up}$ is constant over time and $\beta_{t,down}$ changes linearly over time, then $\mu_{x,t}$ may be factored into one factor $e^{\beta_{t,up}x}$ that depends on age *x* only, not on time *t*, and another factor $\beta_{t,down}e^{-\beta_{t,up}M_t}$ that depends on time *t* only, and not on age *x*. In this case, the analysis used to prove the theorem in Appendix 1 applies immediately (with slightly different expressions for the intercept to allow for the temporal trend in $\beta_{t,down}$). It follows from that analysis that TL describes Gompertz mortality exactly with slope *b* = 2. Hence a temporal trend in $\beta_{t,down}$ cannot explain why the empirical estimates of *b* are strictly less than 2.

Case 2: If $\beta_{t,up}$ changes linearly in time and $\beta_{t,down}$ is constant over time, then $\beta_{t,down}$ has no effect on the slope *b* of TL (though $\beta_{t,down}$ does affect the intercept *a* of TL) because $\beta_{t,down}$ simply rescales the values of $\mu_{x,t}$. If $\beta_{t,up} = s_0 + s_1 t > 0, s_1 \neq 0$ and, as in the theorem, if $M_t = v + w \cdot t > 0, v > 0, w \neq 0,$ for $t = 1, \dots, T$, then

$$\begin{aligned}\beta_{t,up}(x - M_t) &= (s_0 + s_1 t)(x - \{v + w \cdot t\}) \\ &= x(s_0 + s_1 t) - (s_0 + s_1 t)\{v + w \cdot t\} \equiv xf(t) + g(t).\end{aligned}$$

Here $f(t) \equiv s_0 + s_1 t$ is linear in time *t*, and $g(t) \equiv (s_0 + s_1 t)\{v + w \cdot t\}$ is quadratic in time *t*. Then

$$E(\mu_x) = \frac{1}{T}\sum_{t=1}^{T} \mu_{x,t} = \frac{\beta_{t,down}}{T}\sum_{t=1}^{T} \exp\big(xf(t) + g(t)\big),$$

$$\begin{aligned}Var(\mu_x) &= \frac{1}{T}\sum_{t=1}^{T}\left(\mu_{x,t}\right)^2 - \left(E(\mu_x)\right)^2 \\ &= \frac{\left(\beta_{t,down}\right)^2}{T}\sum_{t=1}^{T} \exp 2\big(xf(t) + g(t)\big) - \left(E(\mu_x)\right)^2.\end{aligned}$$

In this case, we are not able to express $\log_{10} Var(\mu_x)$ as a function of $\log_{10} E(\mu_x)$ by means of a simple formula in closed form.

3.3.2.2 Numerical experiment

Instead, we conducted a numerical experiment for women and men of these 12 countries. This numerical experiment provides concrete answers conditional on the observed mortality and may guide possible future mathematical analysis. As an example, we describe our analysis of the observed mortality for Denmark's females from 1960 to 2009.

Input data

A Gompertz model fitted by maximum likelihood to each year's mortality as a function of age yielded time series of estimates of β_t (Figure A-1) and of M_t (Figure A-3) for Danish women. The supplementary spreadsheet gives numerical values. These time series are summarized by the least-squares linear approximations (shown to fewer significant digits in Figures A-1 and A-3),

$$\beta_t = 0.085679264 + 0.00027542975 \cdot t \text{, for } t = 1, \ldots, 50,$$
$$M_t = v + w \cdot t = 80.97560676 + 0.098866512 \cdot t \text{, for } t = 1, \ldots, 50.$$

It seems helpful to appreciate the practical meaning of these two equations. The second equation asserts that Danish females had modal age at death of nearly 81 years in 1960, and that every year thereafter their modal age at death increased by nearly 0.1 year of age per calendar year. According to this regression, in 2009, 50 years after 1960, Danish females had a modal age of death of approximately 86 years (≈ 81 + 50 × 0.1). If the modal age at death increases, why is the rate of increase of mortality with age, β_t, increasing (albeit very slowly)? In the framework of the Gompertz model, the age that matters for mortality (the 'effective age') is not the chronological age x but the excess of the chronological age over the modal age at death, $x - M_t$. As the modal age at death increases 0.1 year of age per calendar year, for each given chronological age x, the effective age $x - M_t$ gets younger by 0.1 year of age per calendar year. Deaths occur at progressively later ages and (because of increasing β_t) mortality rises (slightly) more rapidly (Canudas-Romo 2008, 2010).

Experimental design

In a computational experiment, we put $M_t = v + w \cdot t$ as assumed above. Then we calculated numerically three sets of values of $\mu_{x,t}$ for each age x = 0, ..., 100 and each year t = 1, …, 50. In the first set of values, for the standard Gompertz model with $\beta_{t,down} = \beta_{t,up} = \beta_t$,

$$\mu_{x,t} = \beta_t e^{\beta_t(x-M_t)}.$$

In the second set of values, for the Gompertz model with $\beta_{t,down} = \beta_{25}$ and $\beta_{t,up} = \beta_t$,

$$\mu_{x,t}^{up} = \beta_{25} e^{\beta_t(x-M_t)}.$$

In the third set of values, for the Gompertz model with $\beta_{t,down} = \beta_t$ and $\beta_{t,up} = \beta_{25}$,

$$\mu_{x,t}^{down} = \beta_t e^{\beta_{25}(x-M_t)}.$$

From each set of values, we calculated the corresponding mean and variance of mortality over time for each age x.

Results for all 12 countries

Figures 12 and 13 show the results if $\beta_{t,up} = \beta_{25}$. Figures 14 and 15 show the results if $\beta_{t,down} = \beta_{25}$. Figures 16 and 17 show the log temporal variance as a function of the log temporal mean for these three hypothetical mortality schedules. The results are similar for both sexes in the 12 HMD countries. As an example, we describe the results for Danish women.

Figure 12: TL (solid black line) in fitted female mortality of the model of Gompertz with $\beta_{t,up}$ = constant for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

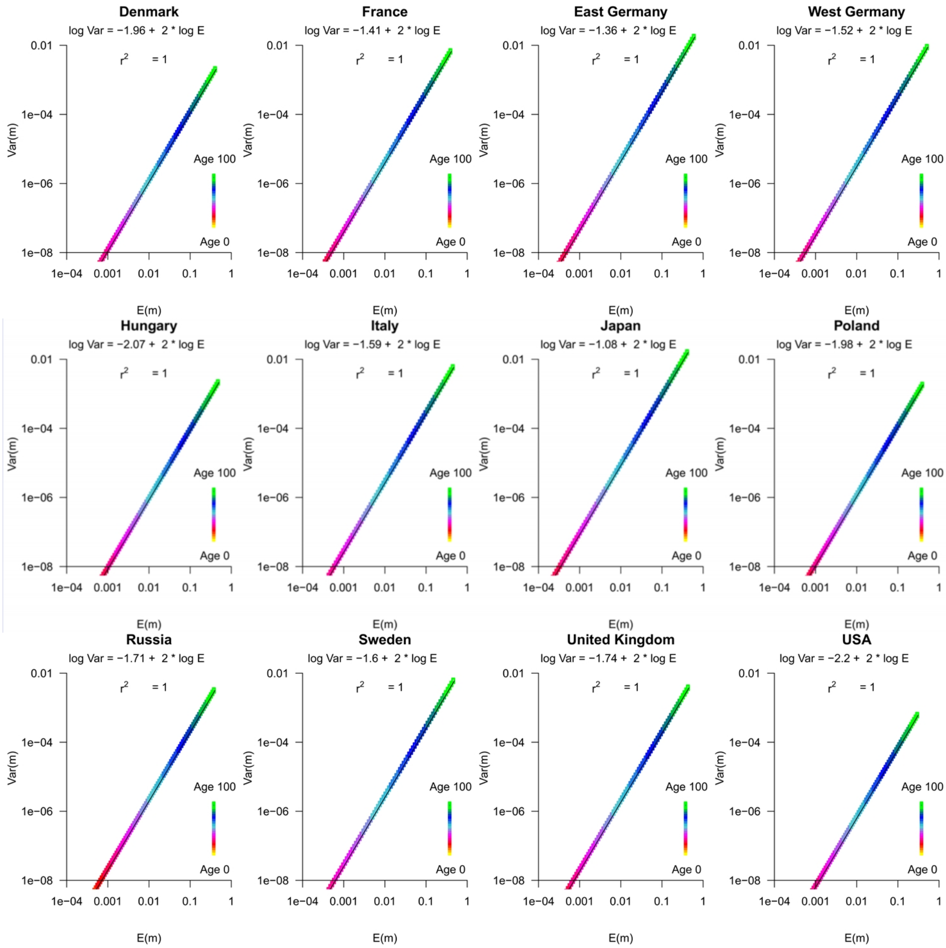


Figure 13: TL (solid black line) in fitted male mortality of the model of Gompertz with $\beta_{t,up}$ = constant for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

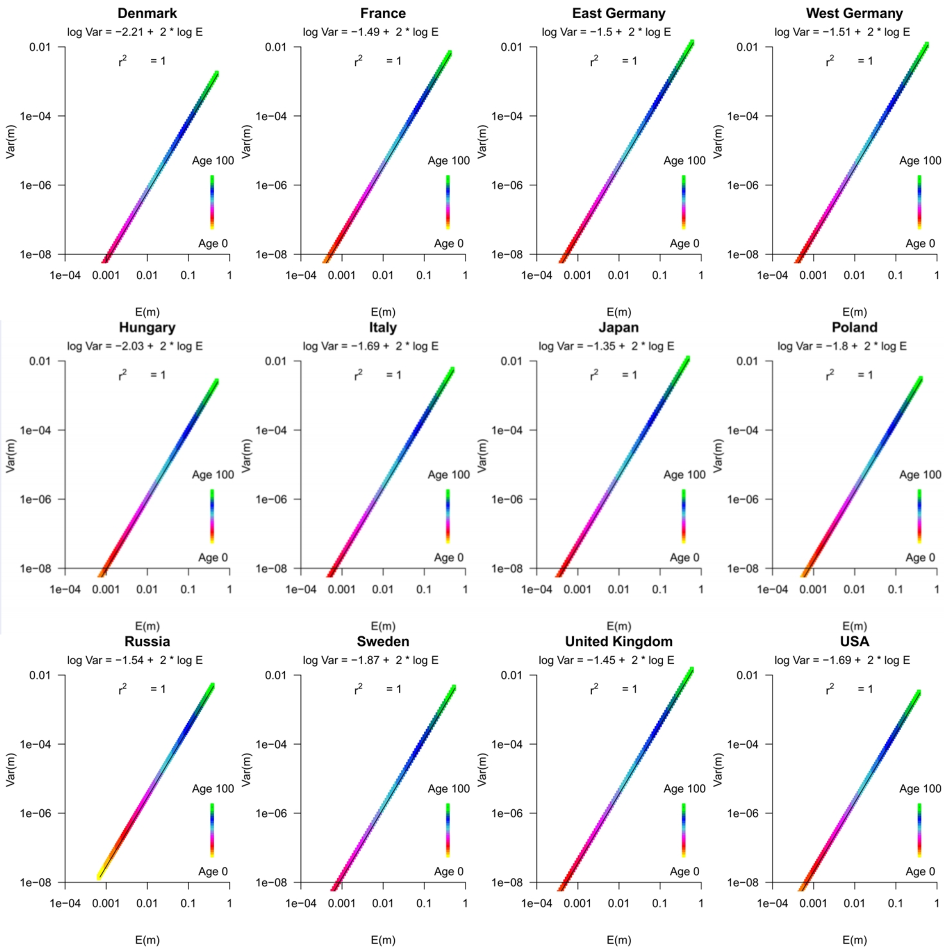


Figure 14: TL (solid black line) in fitted female mortality of the model of Gompertz with $\beta_{t,down}$ = constant for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

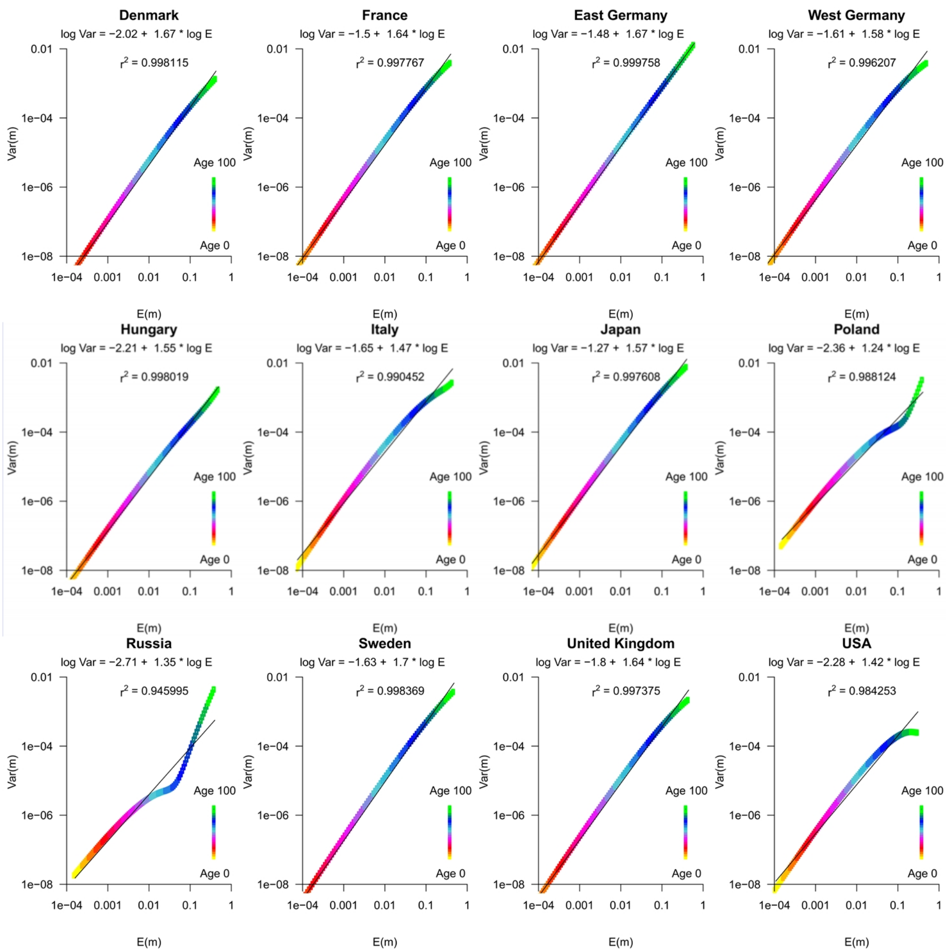


Figure 15: TL (solid black line) in fitted male mortality of the model of Gompertz with $\beta_{t,down}$ = constant for the ages 0 (yellow) to 100 (green) from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

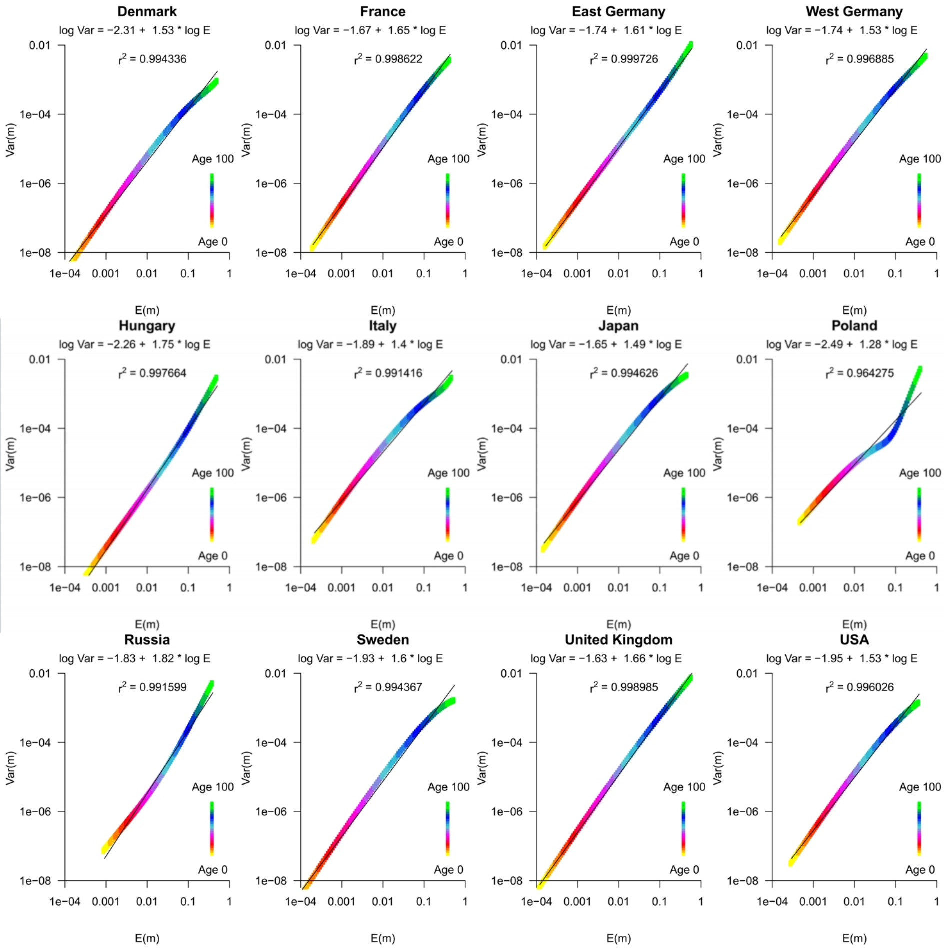


Figure 16: TL (solid dark blue line) in fitted female mortality of the model of Gompertz (solid blue line), of the model of Gompertz with $\beta_{t,up}$ = constant (solid green line) and of the model of Gompertz with $\beta_{t,down}$ = constant (solid red line) for the ages 0 to 100 from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)

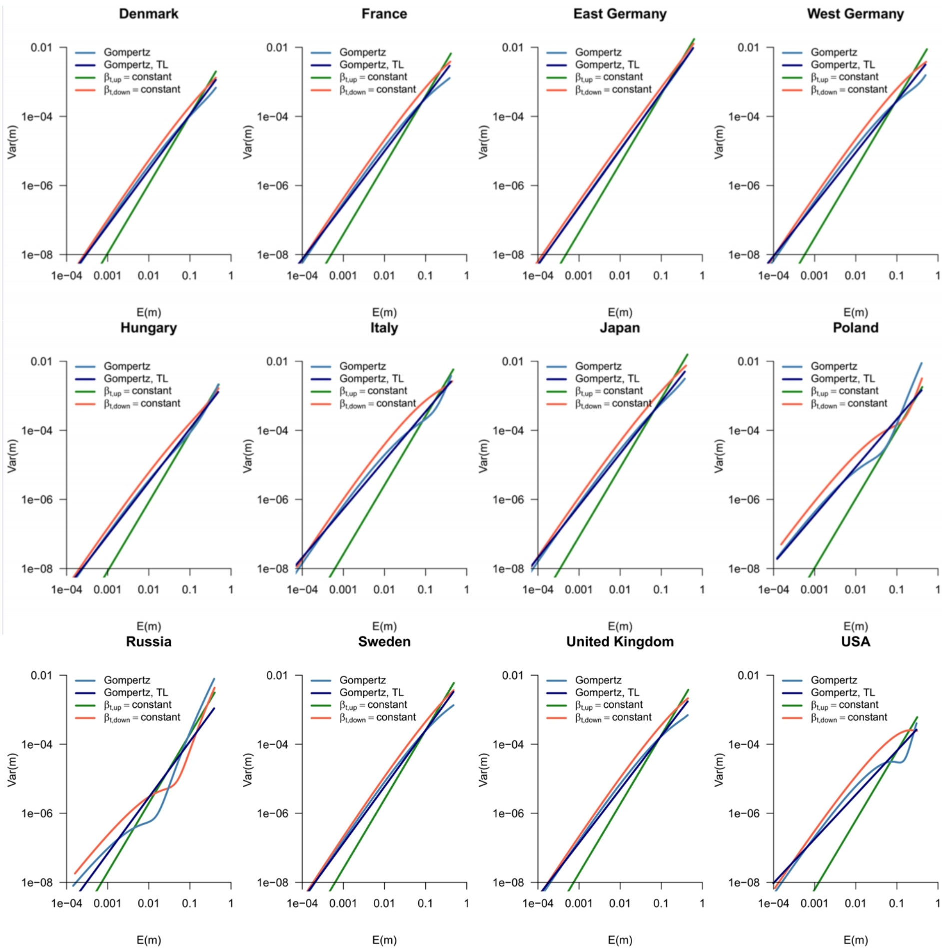
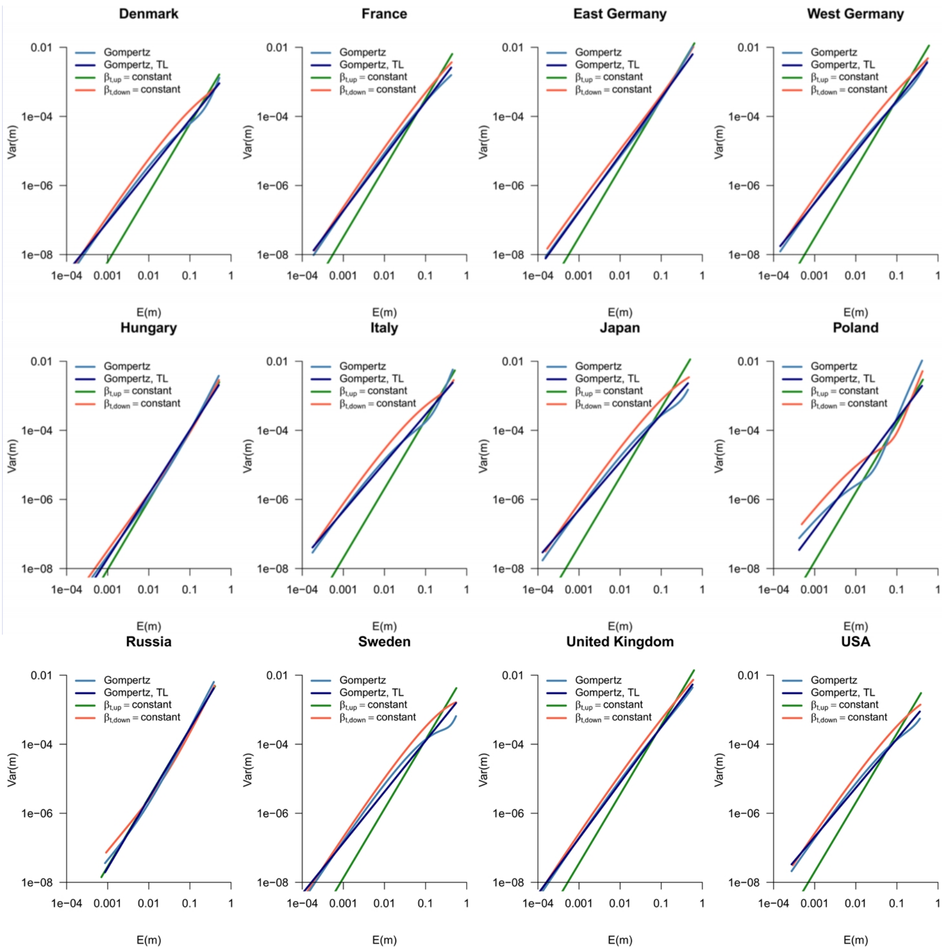


Figure 17: TL (solid dark blue line) in fitted male mortality of the model of Gompertz (solid blue line), of the model of Gompertz with $\beta_{t,up}$ = constant (solid green line) and of the model of Gompertz with $\beta_{t,down}$ = constant (solid red line) for the ages 0 to 100 from 1960 to 2009 on a logarithmic scale (base = 10) for 12 countries of the Human Mortality Database (2015)



For the standard Gompertz model $\mu_{x,t}$ (blue solid line) of Danish women in Figure 16, the relation of log temporal variance to log temporal mean is close to linear (as predicted by TL) except for the large values of the mean and variance of old-age mortality in the upper right corner of the graph. A fitted TL (dark blue solid line, $\log_{10}$ variance = –2.35 + 1.60 $\log_{10}$ mean) approximates the Gompertz log temporal variance and log temporal mean closely over most of their range.

When $\beta_{t,up}$ changes linearly over time and $\beta_{t,down}$ is constant over time, $\mu_{x,t}^{up}$ gives a variance-mean relationship (red solid line) that approximates the Gompertz log variance and log mean closely but is slightly concave (on log–log coordinates). The average slope of this curve, estimated by

$$\log\left(\frac{Var(\mu_{x=100}^{up})}{Var(\mu_{x=0}^{up})}\right) / \log\left(\frac{E(\mu_{x=100}^{up})}{E(\mu_{x=0}^{up})}\right),$$

is 1.64, close to the slope b = 1.60 of the fitted TL. Thus the second case considered above (Gompertz model with $\beta_{t,down} = \beta_{25}$ and $\beta_{t,up} = \beta_t$) gives an approximate explanation of the form and slope of the fitted TL.

By contrast, when $\beta_{t,up}$ is constant over time and $\beta_{t,down}$ changes linearly in time, the relationship of log variance of $\mu_{x,t}^{down}$ to log mean of $\mu_{x,t}^{down}$, calculated numerically (green solid line) from $\mu_{x,t}^{down}$, is visually indistinguishable from linear and has a numerically estimated slope indistinguishable from 2 (to a precision of at least five decimal places). These results confirm numerically the above mathematical analysis of Case 1. This case cannot explain the slope of the TL fitted either to observed mortality or to Gompertz model mortality.

In conclusion, in this example, the linear trend in M_t and the linear trend in $\beta_{t,up}$ in combination explain qualitatively and quantitatively why TL for Gompertz-modeled mortality has slope notably less than 2, unlike the slope of exactly 2 that would be expected theoretically if $\beta_{t,up}$ were constant (regardless of whether $\beta_{t,down}$ is constant or changing in time).

4. Conclusion

4.1 Summary

For females and males in 12 developed countries, the temporal means and temporal variances from 1960 to 2009 of observed age-specific mortality, when one point for each age from 0 to 100 years was plotted on log-log coordinates, fell approximately along a straight line, according to the data of the Human Mortality Database (2015)

(Figures 2 and A-1 in Bohk, Rau, and Cohen 2015; and Figures 3–4 here). This approximate linearity was consistent with Taylor's law. Here we sought to explain this pattern.

We compared TL fitted to temporal means and temporal variances of observed mortality with TL fitted to mortality in the models of Gompertz (1825), Makeham (1860), and Siler (1979, 1983). These models have progressively more parameters and, in the same order, fit the age profile of observed mortality progressively more closely.

We analyzed how well each mortality model's TL matched TL fitted to observed mortality by comparisons of three features: the log–log linearity of the temporal means and temporal variances of the modeled mortality, the age profile (defined as the set of pairs of log(temporal mean mortality at age x) and log(temporal variance of mortality at age x), for all ages x), and the slope.

For log–log linearity, we found that TL approximated mortality in the fitted models of Gompertz, Makeham, and Siler more closely than TL approximated observed mortality. As a consequence, r^2 values of TL of the Gompertz model were very close, and rounded, to 1. Compared to the Gompertz model, the models of Makeham and Siler resulted in closer fits to observed mortality and to the TL of observed mortality. Consequently, the r^2 values of TL of the models of Makeham and Siler were often slightly smaller than those of Gompertz but were also often closer to those of the observed mortality.

For the age profile of TL, we found that the TL of the Siler model fitted to observed mortality had an age profile that was closer to the age profile of TL of observed mortality than were the age profiles of TL of the fitted models of Makeham and Gompertz.

For the slopes of TL, we found that the TL of the Makeham model fitted to observed mortality had a slope that was closest to the slope of TL fitted directly to observed mortality, among the three models. Differences in the slope of TL between males and females in the fitted Makeham models were also closest to the differences in the slope of TL between males and females of observed mortality.

In addition to these empirical and statistical insights, we demonstrated mathematically that the log temporal means and log temporal variances of mortality in the Gompertz model satisfy TL exactly with slope $b = 2$ and an explicitly determined intercept when the modal age at death in the Gompertz model increases linearly with time and the β_t parameter for the increase of mortality with age is constant in time t (or $\beta_{t,up}$ is constant in time). As the Gompertz model is a special case of the more complex models of Makeham and Siler, these theoretical findings also apply to certain parameter values of the other two models.

Empirically, however, the slopes of TL fitted to observed mortality ranged from 1.65 to 1.87 and the slopes of TL for the mortality models were all at least 1.28 and smaller than 2 (apart from the two exceptions noted above, for Gompertz model mortality of Russian males and Siler model mortality of French females). To explain why the slopes of TL fitted to observed mortality and the fitted models were notably smaller than 2 (with the two exceptions just noted), our computational experiments showed that, in the presence of a linearly increasing modal age at death, it was necessary and sufficient to take into account in the Gompertz model a linear trend in $\beta_{t,up}$. When $\beta_{t,up}$ increased linearly in time, the slope of TL based on Gompertz mortality was less than 2, and when $\beta_{t,up}$ was constant, the slope of TL based on Gompertz mortality was numerically (and mathematically) indistinguishable from 2. We tested numerically and confirmed this explanation for women and men in 12 countries of the Human Mortality Database. These numerical results indicate that, as long as $\beta_{t,up}$, the growth rate of mortality with age, increases linearly with time, TL fitted to mortality will have a slope that is not equal to 2.

To conclude, our empirical, statistical, mathematical, and numerical findings confirm that the temporal TL is a regular pattern rooted in widely recognized models of the age pattern and temporal evolution of human mortality.

4.2 Future research

These results raise further theoretical and empirical questions.

Our mathematical analysis of the Gompertz model remains incomplete when both parameters (the modal age at death and the growth rate of mortality with age) change in time. Our computational experiment gave clear results about this case, but we have not proved these results mathematically. Mathematical analysis is needed to reveal the necessary and sufficient conditions for TL fitted to Gompertz mortality to have a slope less than 2 (not merely different from 2).

It would be desirable to complete the mathematical analysis of the Gompertz model and to extend it to the Makeham, Siler, and other more complex models, for example, those of Heligman and Pollard (1980) and Thiele (1872), and piecewise constant mortality models of, for example, Brouhns, Denuit, and Vermunt (2002) and Cairns et al. (2009). These models may provide more precise approximations to empirical age profiles of mortality. However, their larger number of parameters and their greater mathematical complexity make them more difficult to analyze mathematically and to understand. Since our goal here was to understand an empirical pattern in a transparent way, we focused on simpler mortality models.

Future research may extend the analysis to still more complex models. A potentially productive approach to analyzing temporal trends and variations in mortality would be to construct a generalized linear model (GLM) of all the observed mortality rates simultaneously, as in Brouhns et al. (2002), Cairns et al. (2009), and Renshaw and Haberman (2006), and reviewed by Booth and Tickle (2008). In a GLM approach, the dependent variable would be $\mu_{x,t}$ for all ages x, all years t, both sexes, and all countries. The independent variables (predictors) would be age x, year t, sex (female or male), country, and various higher-order (e.g., x^2 and t^2 to model curvature) and interaction terms to be determined in the course of the analysis. The coefficients of predictor t and t^2 would quantify the importance of systematic trends, linear and nonlinear respectively. A GLM can estimate the mean and the variance of $\mu_{x,t}$ simultaneously (for example, by using quasi-likelihood techniques for the variance). With the estimates of means and variances of $\mu_{x,t}$ from a GLM, it would be possible to test TL with finer resolution than has been possible with the traditional approach used here, in which temporal means and temporal variances are computed independently for each age, sex, and country.

A GLM could also be used to analyze mortality from each of the three models considered here, and the structure and coefficients of the GLM for modeled mortality could be compared with the structure and coefficients of the GLM for observed mortality. This comparison would permit an evaluation of the models with finer resolution than has been possible with the traditional approach used here.

Testing TL in deterministic mortality models is a special case of applying TL to smoothed data, with some of the initial variability removed, leaving only dominant trends. Here the ‘smoothed data’ are the predictions of the models. Comparison of the goodness of fit and parameter estimates of TL with such smoothed data versus with the original data shows whether the smoothed trends or the variability about those trends dominate the goodness of fit and parameter estimates of TL. In the examples in this paper, because the TL fitted to models is generally close to the TL fitted to the original mortality observations, it is clear that the smoothed trends play the dominant role in the success of TL. Further research is needed to show the conditions under which the smoothed data versus the fluctuations around trends dominate the performance of TL.

Another empirical question and approach prompted by a reviewer’s question is this. For any fixed age x, observed mortality $\mu_{x,t}$ over the 50 years t = 1960, ..., 2009 may have a systematic trend, fluctuations from this trend in each year t, and an interaction between the trend and the fluctuations (e.g., temporal heteroscedasticity or temporal changes in the skewness of fluctuations). In future research, it would be interesting to decompose the temporal mean and the temporal variance of observed mortality at a given age into the contributions due to a systematic trend in time, fluctuations, and their interaction; and to decompose the overall temporal TL of mortality into components arising from trend, fluctuations, and interaction. A parallel theoretical analysis could decompose age-specific mortality from models that explicitly incorporated stochastic fluctuations in mortality, unlike the Gompertz, Makeham, and Siler models. This empirical investigation could provide a foundation for new theory generalizing the Gompertz and other models to allow for stochastic fluctuations in mortality over time.

Reviewer Hal Caswell posed a more general theoretical question that is also related to variation in mortality. Temporal fluctuations in mortality rates are a component of a demographic model in a stochastic environment. What are the consequences for stochastic population growth of greater temporal variance in mortality at (older) ages where the mean mortality is also higher? This question shows the potential use of TL applied to mortality in modeling and simulating stochastic age-structured populations.

The above outlines of potential applications of TL in human mortality indicate TL's possible usefulness and relevance in formal and empirical demographic research.

5. Acknowledgments

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Appendix 1: Taylor's law with slope 2 describes the Gompertz model

Theorem

The Gompertz mortality model with modal age at death increasing linearly in time obeys a cross-age-scenario of Taylor's law (TL) exactly with slope $b = 2$. Explicitly, assuming the Gompertz model $\mu_{x,t}$ at age *x* and time *t*,

$$\mu_{x,t} = \beta_t e^{\beta_t (x - M_t)}, \beta_t > 0, M_t > 0, \text{for } t = 1, \dots, T, x = 1, \dots, X,$$

with a linear change (increase or decrease) over time in the modal age at death,

$$M_t = v + w \cdot t > 0, v > 0, w \neq 0, \text{for } t = 1, \dots, T,$$

and an exponential rate increase of mortality with age *x* that is constant over time *t*,

$$\beta_t = \beta > 0, \text{for } t = 1, \dots, T,$$

then TL holds with slope 2 and intercept $\log(K_2 - K_1^2) - 2\log K_1$ on log–log coordinates:

$$\log Var(\mu_x) = \log\left(\frac{K_2 - K_1^2}{K_1^2}\right) + 2 \cdot \log E(\mu_x), \text{for } x = 1, \dots, X,$$

where the positive constants K_1, K_2 are defined below.

Proof

From the assumptions,

$$\mu_{x,t} = \beta_t e^{\beta_t (x - M_t)} = \beta e^{\beta (x - \{v + w \cdot t\})} = \beta e^{\beta (x - v)} e^{-\beta w t},$$

which implies that $\mu_{x,t}$ is an exponentially increasing function of age *x* for every time *t*. It also implies that $\mu_{x,t}$ is an exponentially decreasing function of time *t* for every age *x* if $w > 0$, and is an increasing function of time *t* for every age *x* if $w < 0$. Then, using the definitions in the main text of $E(\mu_x)$ as the temporal mean and $Var(\mu_x)$ as the temporal variance of mortality at age *x*,

$$E(\mu_x) = \frac{1}{T}\sum_{t=1}^{T}\mu_{x,t} = \frac{1}{T}\beta e^{\beta(x-v)}\sum_{t=1}^{T} e^{-\beta wt} = e^{\beta x}\left[\frac{\beta e^{-\beta v}}{T}\sum_{t=1}^{T} e^{-\beta wt}\right],$$

where the first factor $e^{\beta x}$ varies with age x only and the bracketed second factor varies with time t and T only. Define

$$K_1 \equiv \frac{\beta e^{-\beta v}}{T}\sum_{t=1}^{T} e^{-\beta wt}, \qquad q \equiv e^{-\beta w}.$$

K_1 does not depend on age x. Then since $\beta > 0, w \neq 0$, we have $q < 1$ if $w > 0$ and $q > 1$ if $w < 0$ and in both cases

$$K_1 = \frac{\beta e^{-\beta v}}{T}(q^1 + q^2 + \cdots + q^T) = \frac{\beta e^{-\beta v}}{T} q(1 + q + \cdots + q^{T-1})$$
$$= \frac{\beta e^{-\beta v}}{T} \cdot \frac{q(1 - q^{T-1})}{1 - q}.$$

Since $K_1 > 0$, $E(\mu_x) = K_1 e^{\beta x}$ increases exponentially at rate β with increasing x. Also

$$Var(\mu_x) = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_{x,t} - E(\mu_x)\right)^2 = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_{x,t}\right)^2 - \left(E(\mu_x)\right)^2$$
$$= \frac{1}{T}\sum_{t=1}^{T}\left(\beta e^{\beta(x-v)} e^{-\beta wt}\right)^2 - \left(K_1 e^{\beta x}\right)^2$$
$$= e^{2\beta x}\left[\frac{\beta^2 e^{-2\beta v}}{T}\right]\sum_{t=1}^{T} e^{-2\beta wt} - K_1^2 e^{2\beta x}.$$

Define

$$K_2 \equiv \left[\frac{\beta^2 e^{-2\beta v}}{T}\right]\sum_{t=1}^{T} e^{-2\beta wt}.$$

K_2 does not depend on age x. Then

$$K_2 = \left[\frac{\beta^2 e^{-2\beta v}}{T}\right](q^{2\cdot 1} + q^{2\cdot 2} + \cdots + q^{2\cdot T}) = \left[\frac{\beta^2 e^{-2\beta v}}{T}\right]\frac{q^2(1 - q^{2\cdot(T-1)})}{1 - q^2}.$$

Also

$$Var(\mu_x) = (K_2 - K_1^2)e^{2\beta x}.$$

By Cauchy's inequality, $K_2 - K_1^2 > 0$. Therefore $Var(\mu_x)$ increases exponentially at rate 2β with increasing age x. Thus, we showed that

$$E(\mu_x) = K_1 e^{\beta x},$$
$$Var(\mu_x) = (K_2 - K_1^2)e^{2\beta x}.$$

Therefore

$$Var(\mu_x) = \frac{K_2 - K_1^2}{K_1^2}\left(E(\mu_x)\right)^2.$$

Appendix 2: Taylor's law with slope less than 2 describes the model of Makeham

In the Makeham model, eq. (2), the additivity of expectations yields

$$E_M(\mu_x) = E(c_t) + E\left(\beta_t e^{\beta_t(x-M_t)}\right) = E(c_t) + E_G(\mu_x).$$

The subscript *M* denotes the Makeham model, and the subscript *G* denotes the Gompertz model. Calculating the variance in the Makeham model requires specifying the relation between c_t and $\beta_t e^{\beta_t(x-M_t)}$. Based on the parameter estimates of the Makeham model in Figures A-5–A-10, we examine this empirically plausible special case:

$$c_t = ce^{-dt}, c > 0, d > 0,$$
$$\beta_t = \beta > 0,$$
$$M_t = v + w \cdot t > 0, v > 0, w > 0.$$

Thus

$$\mu_{x,t} = ce^{-dt} + e^{\beta x}\beta e^{-\beta\{v + w\cdot t\}}.$$

On the right side, the first term depends only on t, not on x, and the second term factors into one factor $e^{\beta x}$ that depends on x only and another factor that depends on t only. By summing geometric series as in Appendix 1, we can get explicit expressions for the constant C_1 (and K_1 is identical to that constant in the Gompertz model) in the expression

$$E_M(\mu_x) = C_1 + K_1 e^{\beta x}.$$

With increasing age x, $E_M(\mu_x)$ grows as the factor $e^{\beta x}$. Also,

$$\mu_{x,t}^2 = c^2 e^{-2dt} + e^{2\beta x}\beta^2 e^{-2\beta\{v+w\cdot t\}} + 2ce^{\beta x}\beta e^{-\beta\{v+w\cdot t\}-dt}.$$

Therefore

$$\begin{aligned} E_M\left(\mu_{x,t}^2\right) &= C_2 + K_3 e^{2\beta x} + K_4 e^{\beta x}, \\ E_M^2(\mu_x) &= C_1^2 + K_1^2 e^{2\beta x} + 2C_1 K_1 e^{\beta x}, \\ Var_M(\mu_x) &= E_M\left(\mu_{x,t}^2\right) - E_M^2(\mu_x) \\ &= C_2 - C_1^2 + (K_3 - K_1^2)e^{2\beta x} + (K_4 - 2C_1K_1)e^{\beta x}. \end{aligned}$$

The same elementary methods used in Appendix 1 can determine K_3 and K_4 explicitly.

From Figures A-7–A-8, $\beta_t \approx 0.1$ for both females and males. Hence as x increases from 0 to 100, $e^{\beta x}$ increases from $e^0 = 1$ to approximately $e^{10} \approx 2.2 \times 10^4$. Hence the term of $Var_M(\mu_x)$ that contains the factor $e^{2\beta x}$, which is approximately $e^{20} \approx 4.8 \times 10^8$ when $x = 100$, increasingly dominates the term of $Var_M(\mu_x)$ that contains the factor $e^{\beta x}$. So, to a first approximation, neglecting all but the dominant terms, $Var_M(\mu_x)$ scales with increasing age x as the square of $E_M(\mu_x)$. Thus, asymptotically for increasing age x, TL holds approximately with slope $b \approx 2$.

This is only a first approximation. $E_M(\mu_x)$ and $Var_M(\mu_x)$ each contains a constant term, and $Var_M(\mu_x)$ contains a term with factor $e^{\beta x}$, which scales more slowly than the dominant term of $Var_M(\mu_x)$ that contains the square of $e^{\beta x}$. Thus $Var_M(\mu_x)$ scales with increasing x more slowly than the square of $E_M(\mu_x)$, that is, TL is expected to hold approximately with a slope less than 2. In all 12 countries, the estimated values of the TL slope never exceeded the Japanese record of $b = 1.86$ for females (Figure 7) and never exceeded the Japanese and French records of $b = 1.82$ for males (Figure 8). Both record values were substantially less than 2. Thus, we have given, in this special case, an argument to explain why TL with slope less than 2 approximates the mortality of the Makeham model reasonably well.

Figure A-1: Annual estimates for β of the Gompertz model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

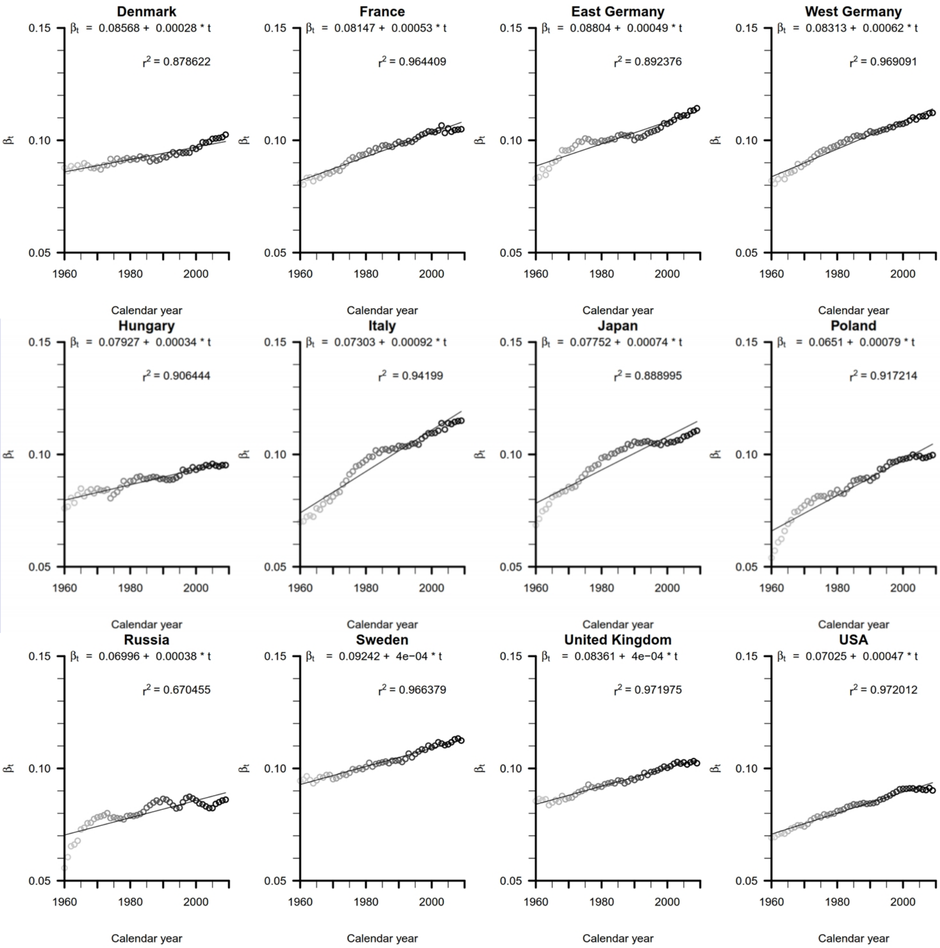

Figure A-2: Annual estimates for β of the Gompertz model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

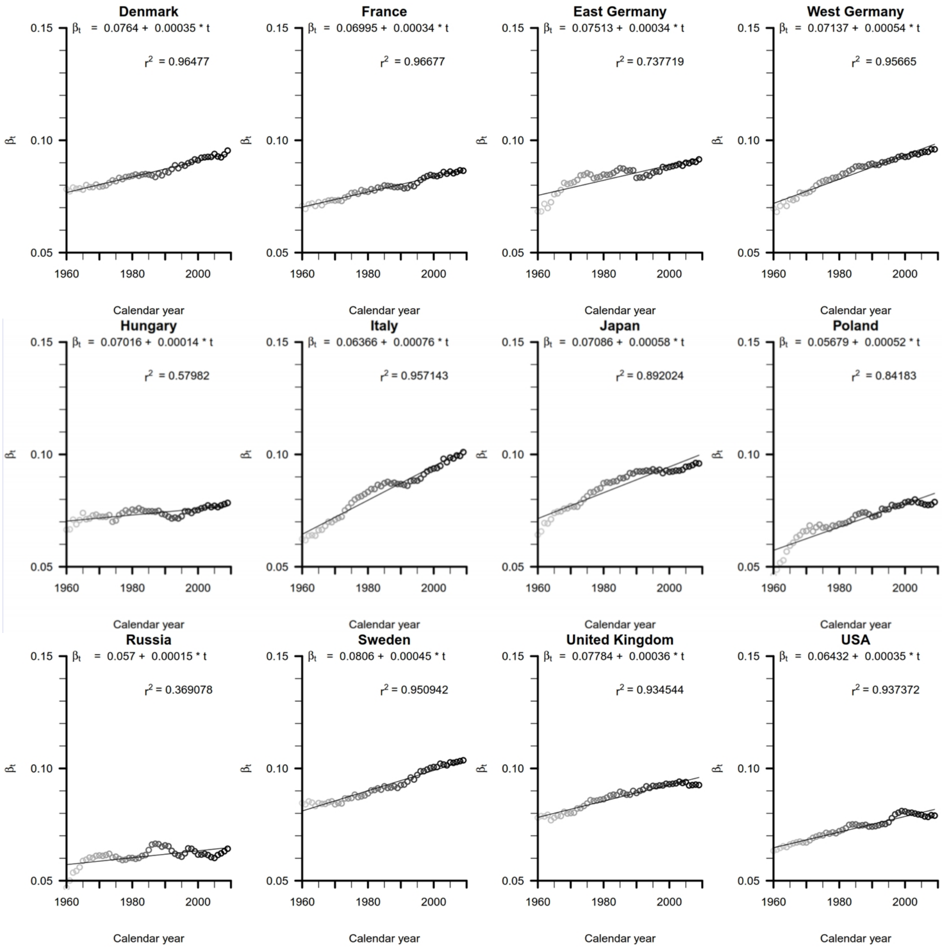

Figure A-3: Annual estimates for M of the Gompertz model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

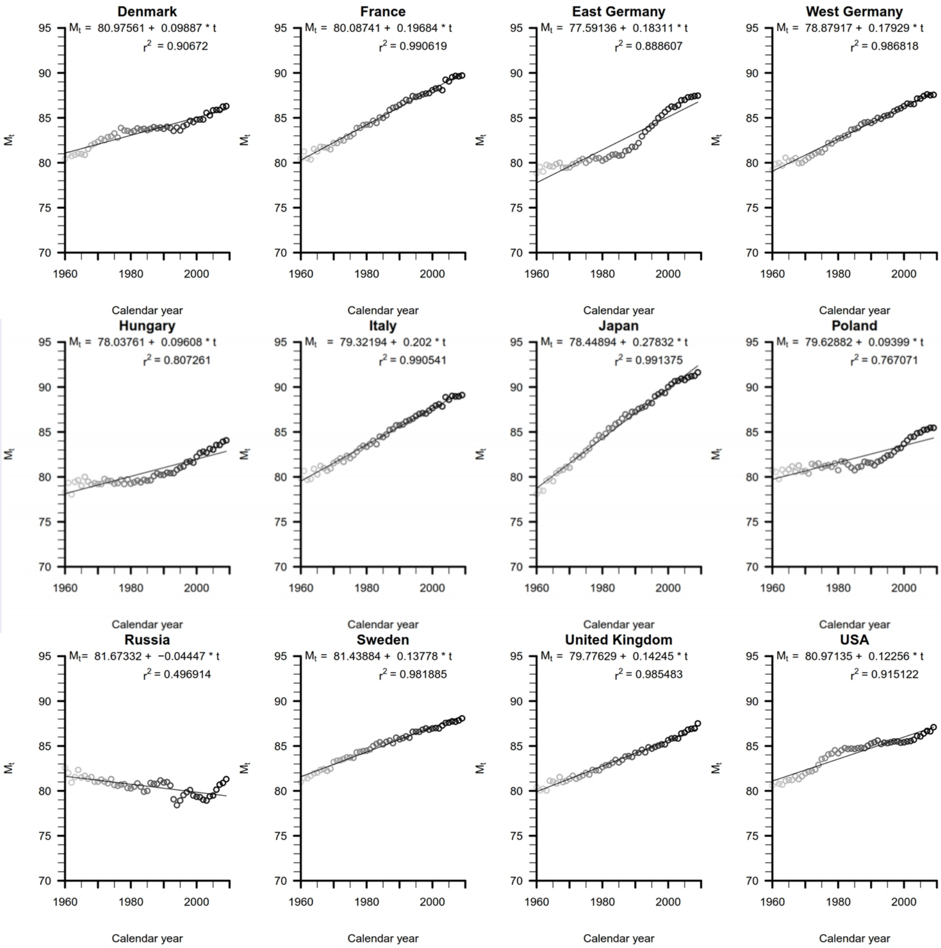

Figure A-4: Annual estimates for M of the Gompertz model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

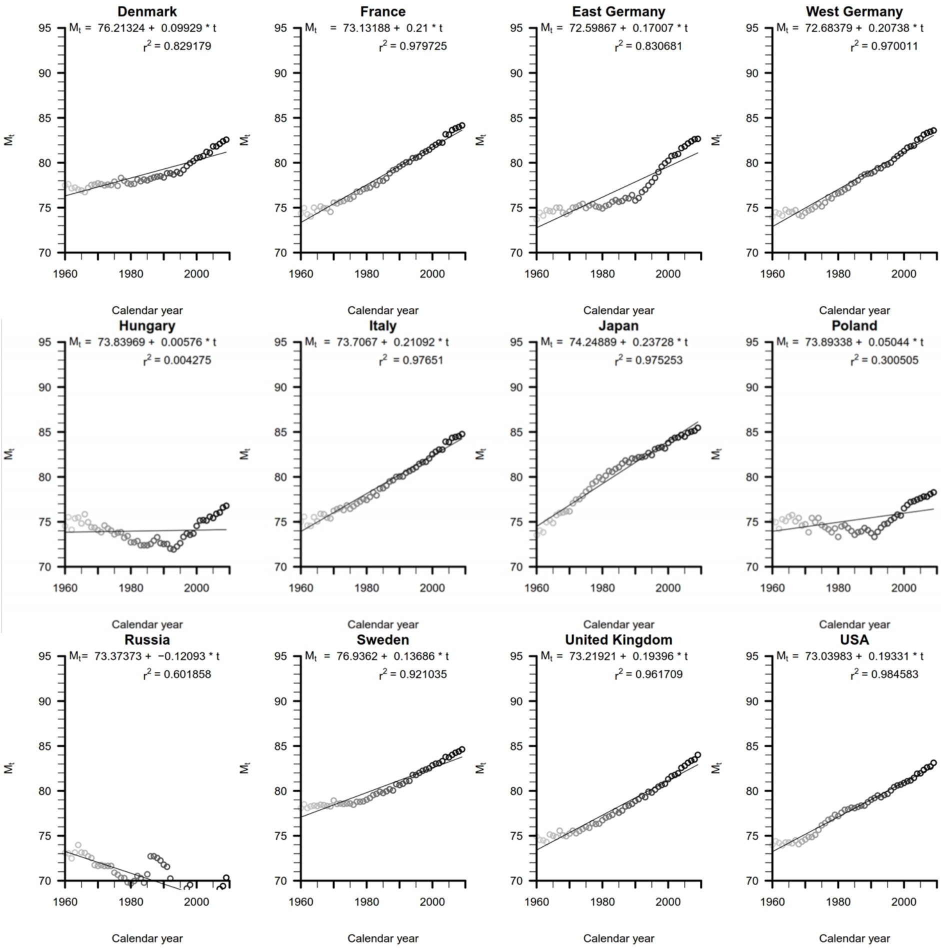

Figure A-5: Annual estimates for c of the Makeham model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

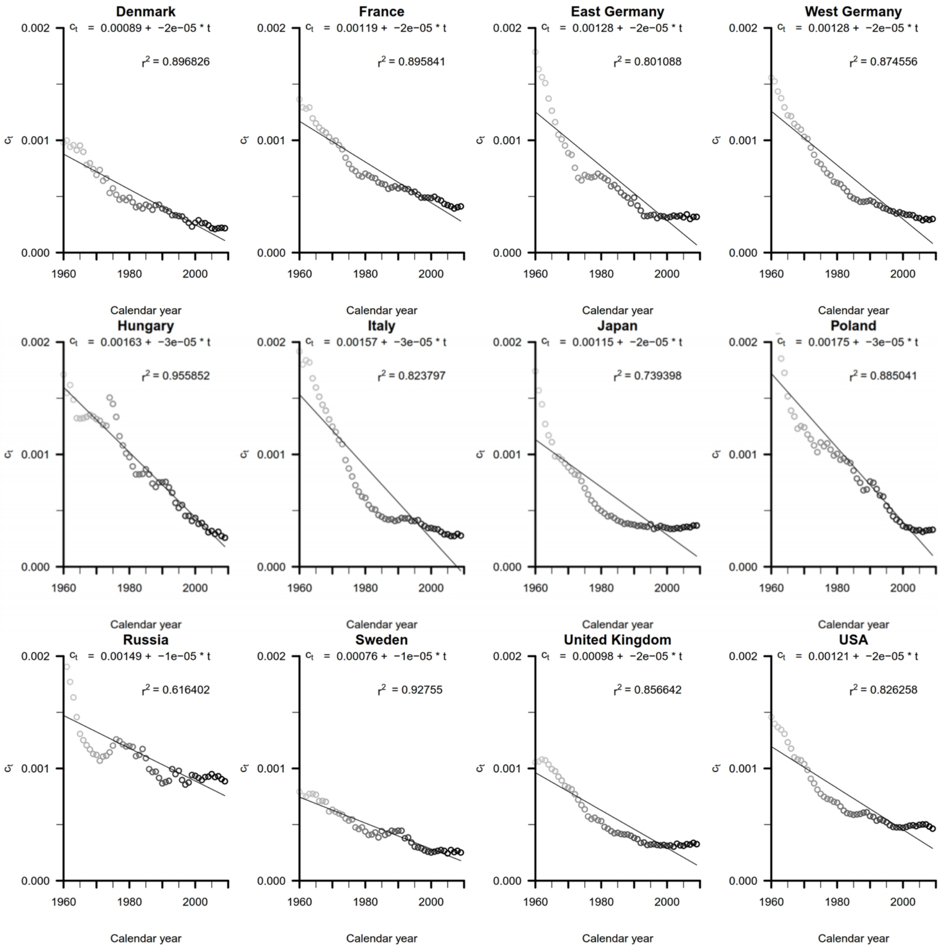

Figure A-6: Annual estimates for c of the Makeham model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

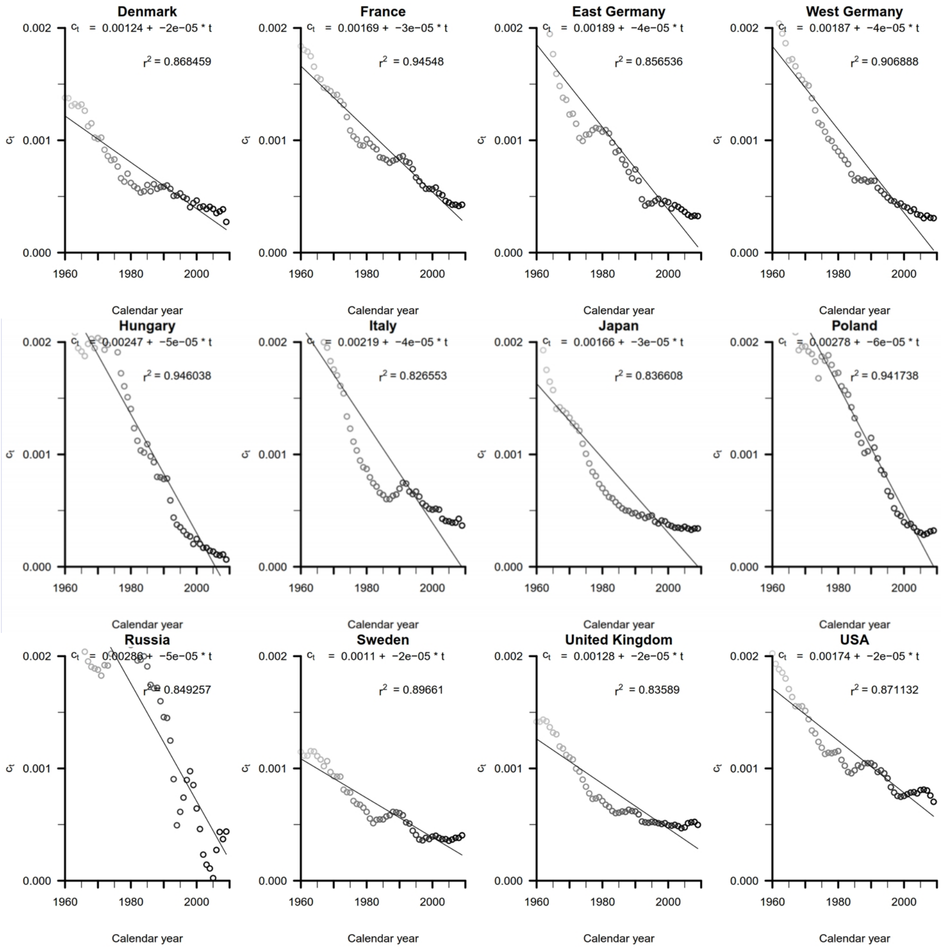

Figure A-7: Annual estimates for β of the Makeham model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

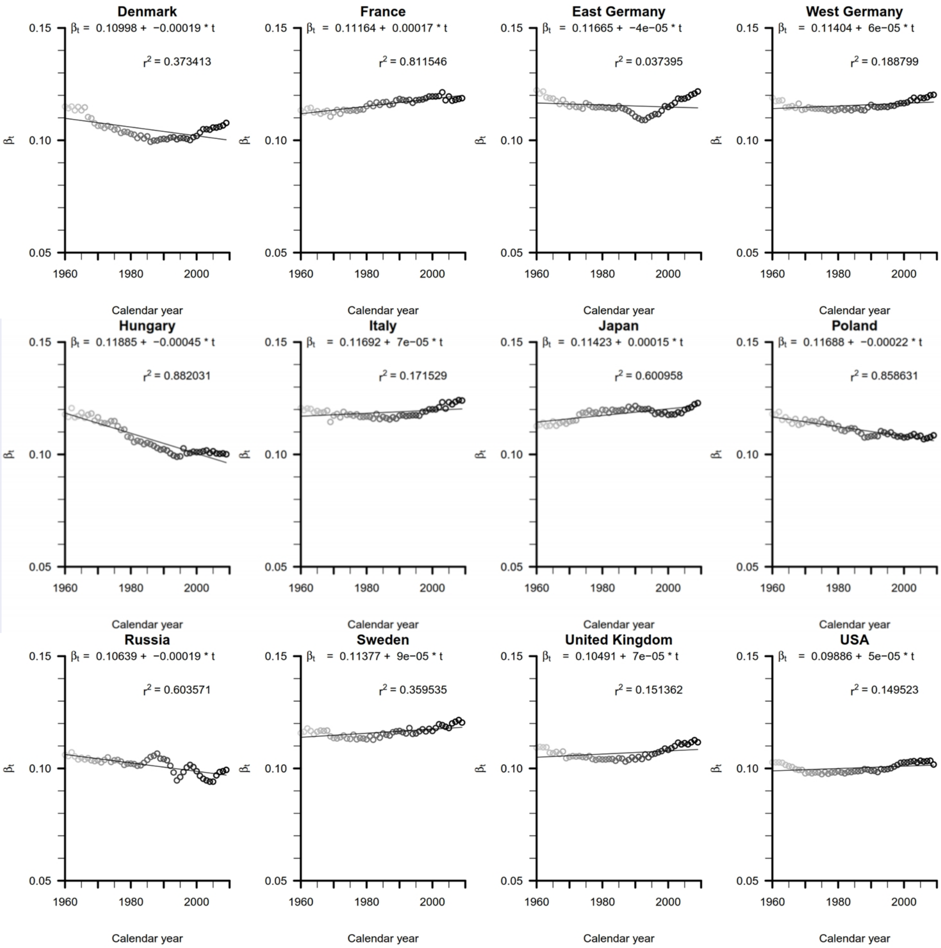

Figure A-8: Annual estimates for β of the Makeham model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

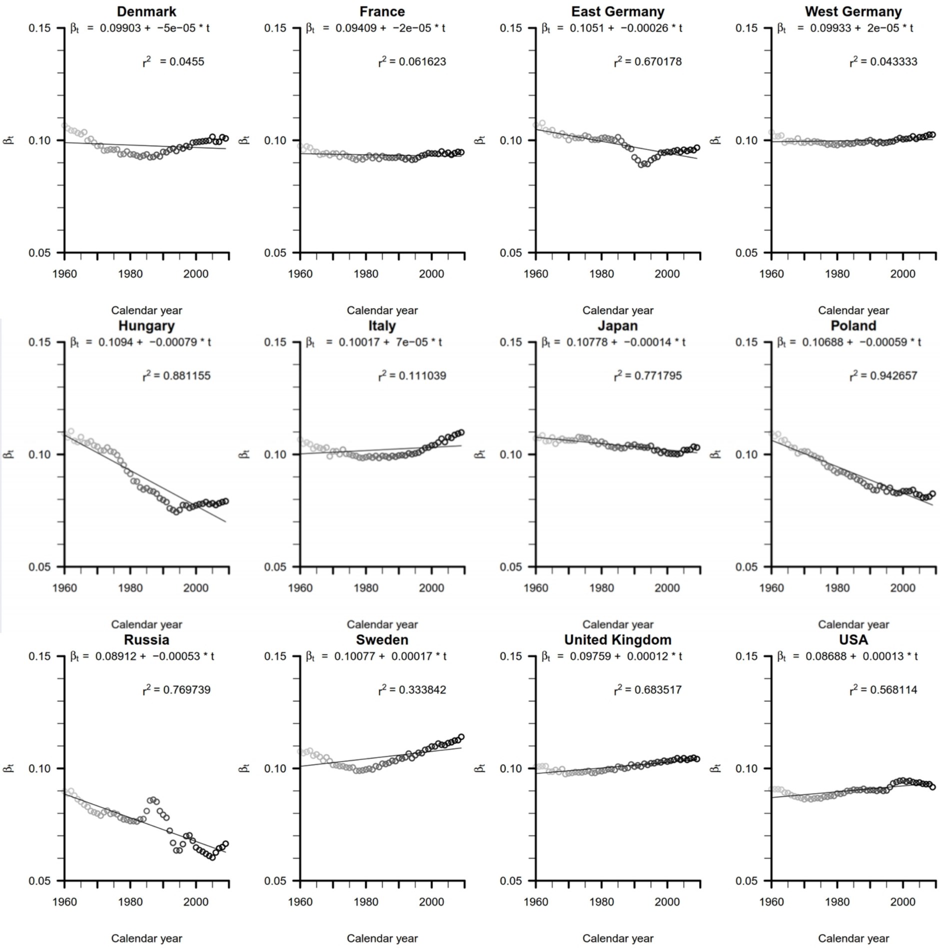

Figure A-9: Annual estimates for M of the Makeham model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

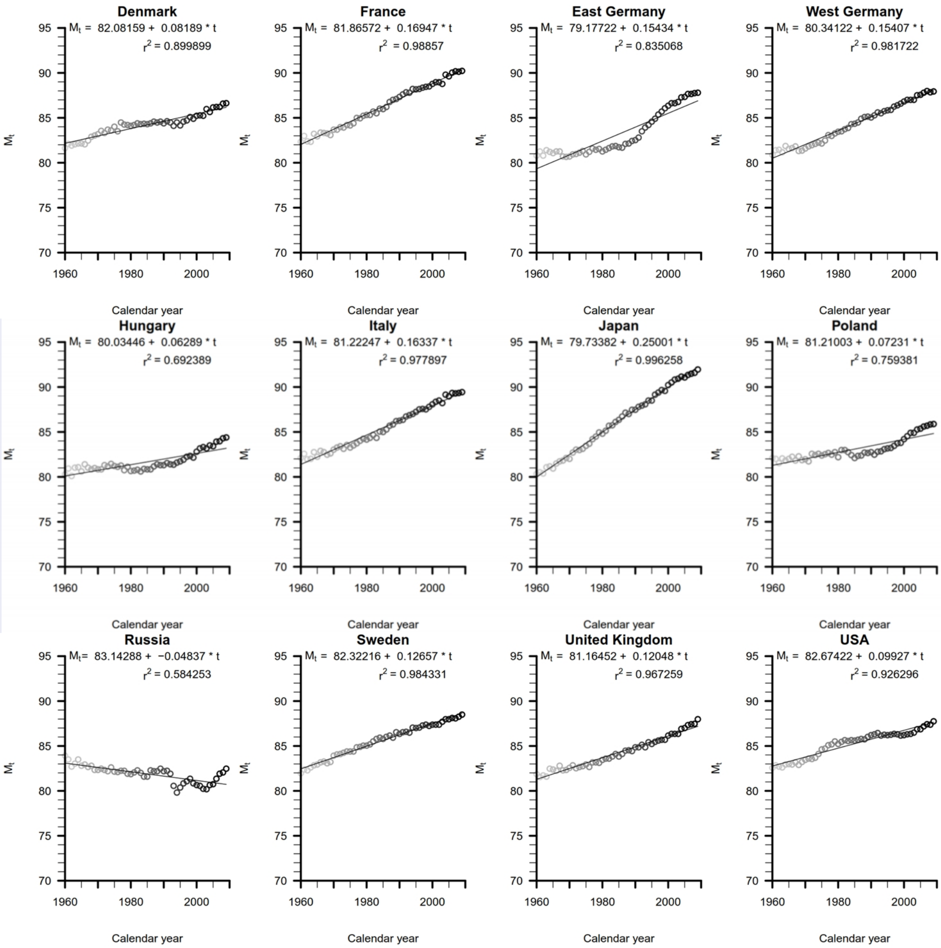

Figure A-10: Annual estimates for M of the Makeham model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

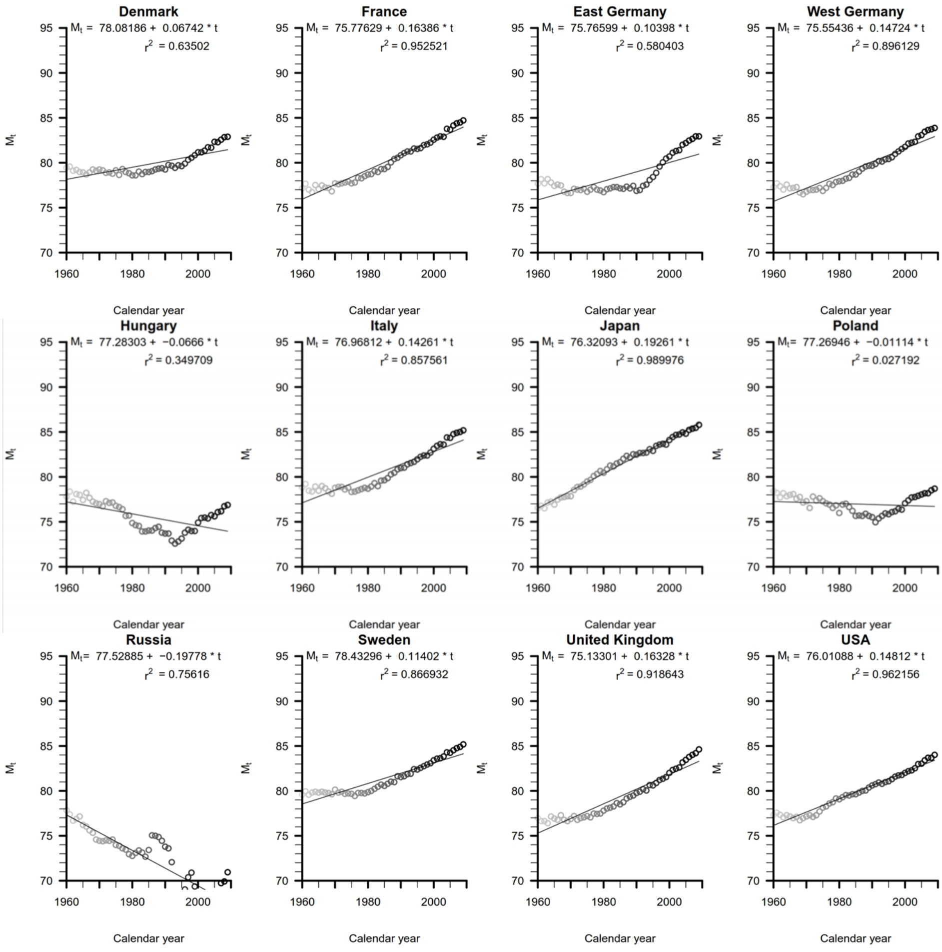

Figure A-11: Annual estimates for α of the Siler model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

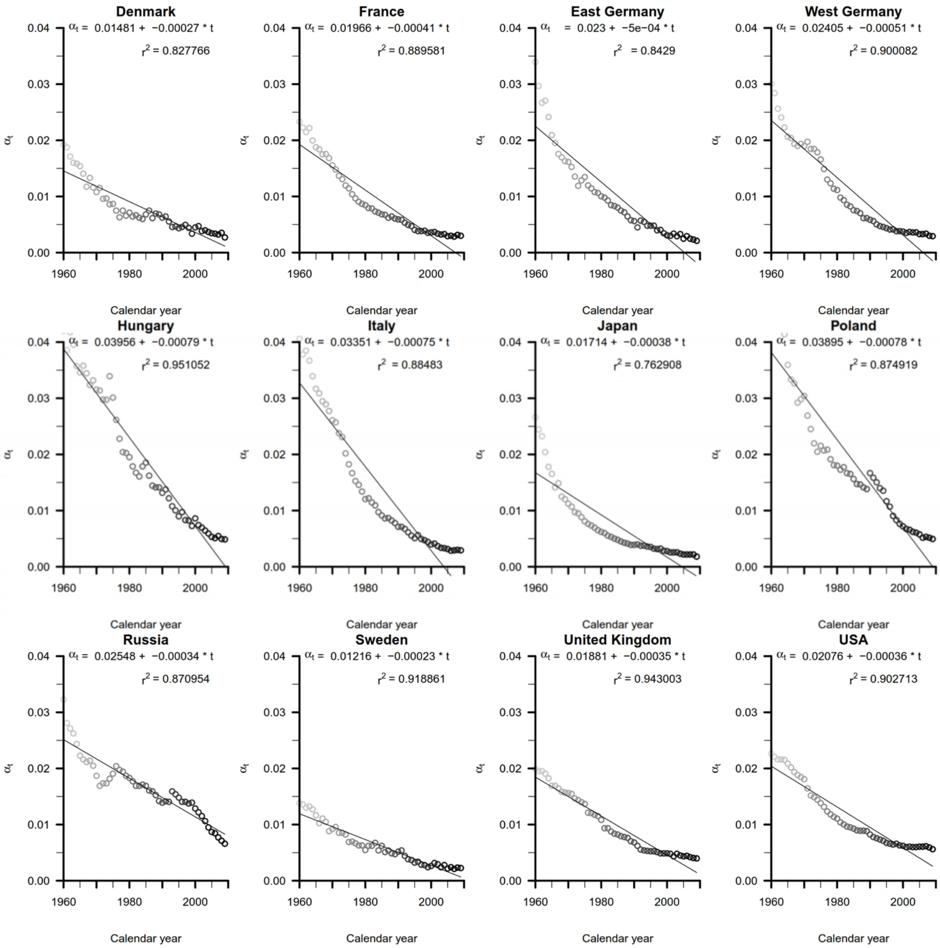

Figure A-12: Annual estimates for α of the Siler model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

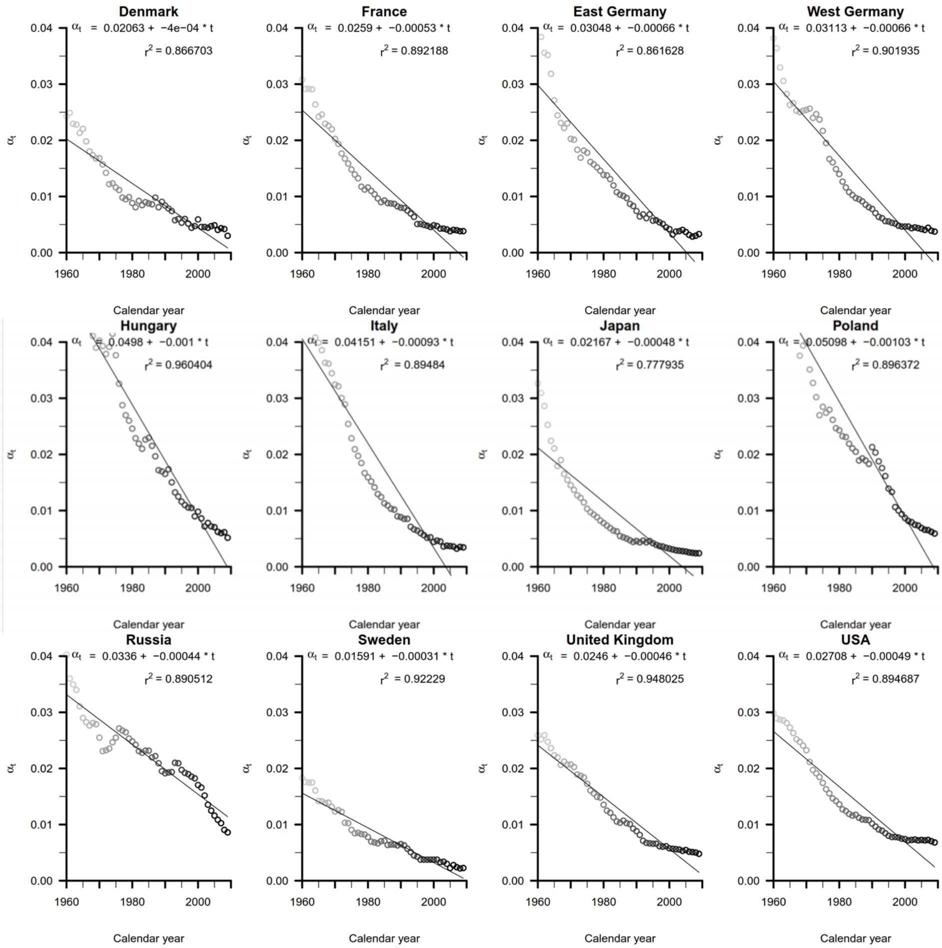

Figure A-13: Annual estimates for β_1 of the Siler model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

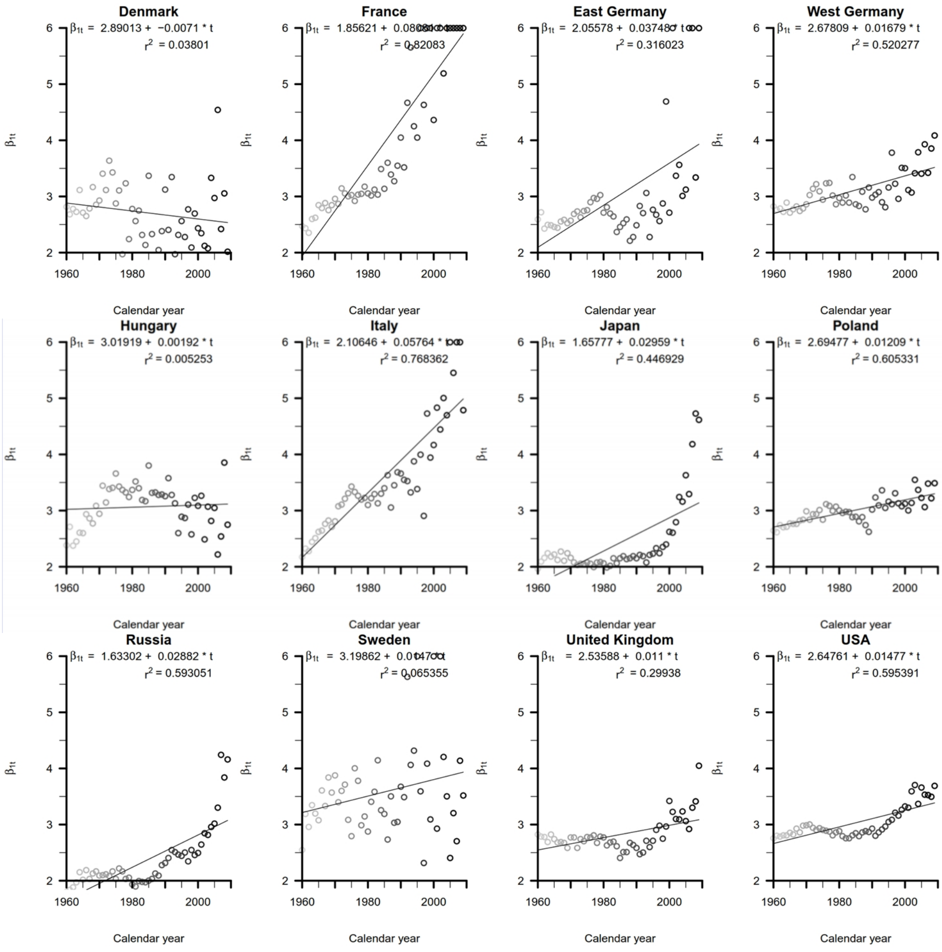

Figure A-14: Annual estimates for β_1 of the Siler model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

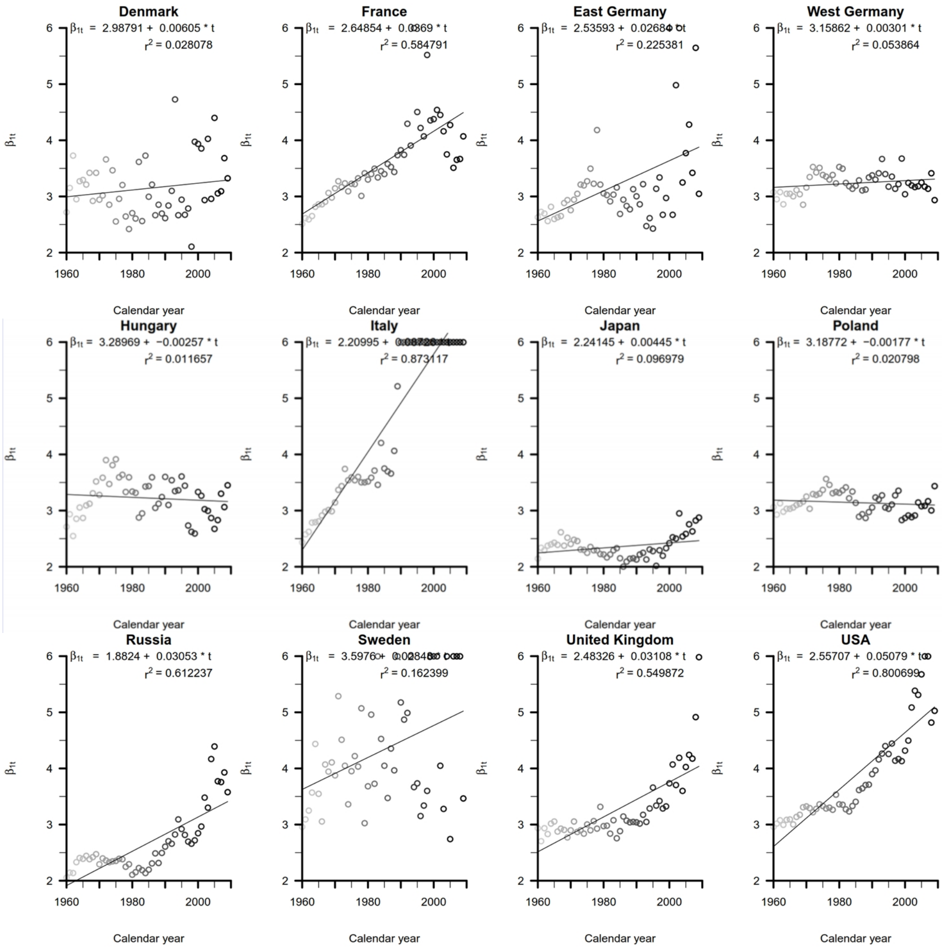

Figure A-15: Annual estimates for c of the Siler model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

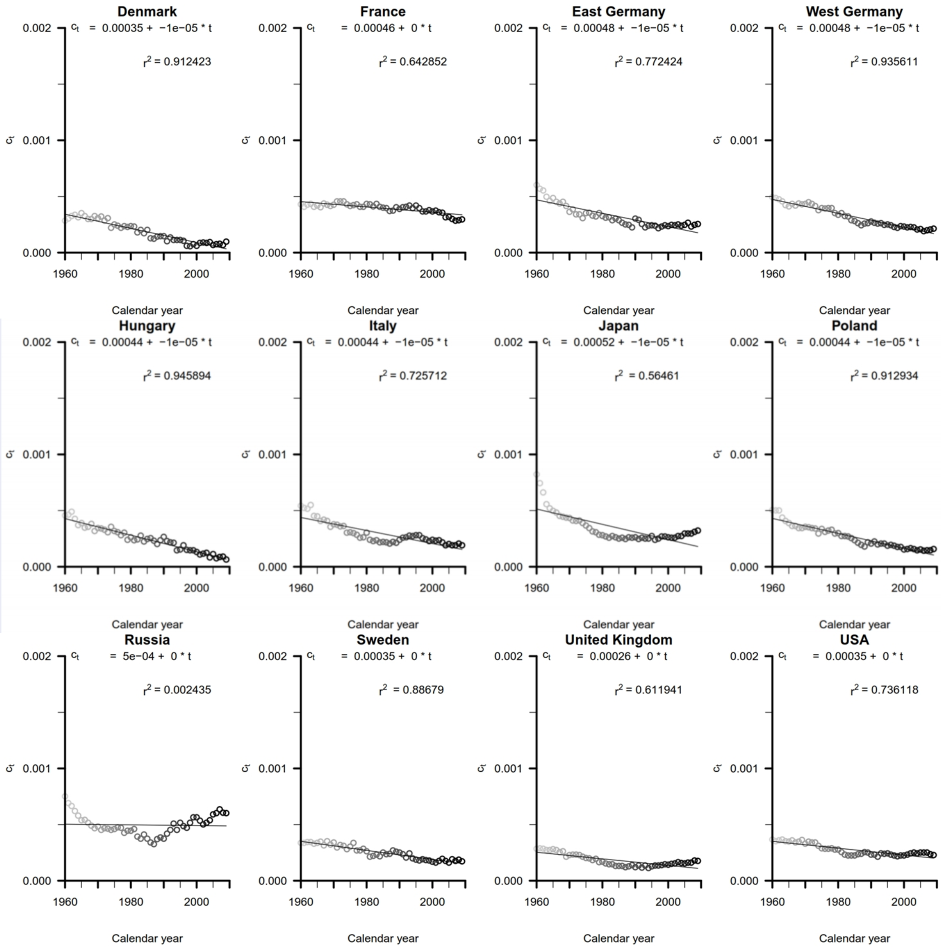

Figure A-16: Annual estimates for c of the Siler model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

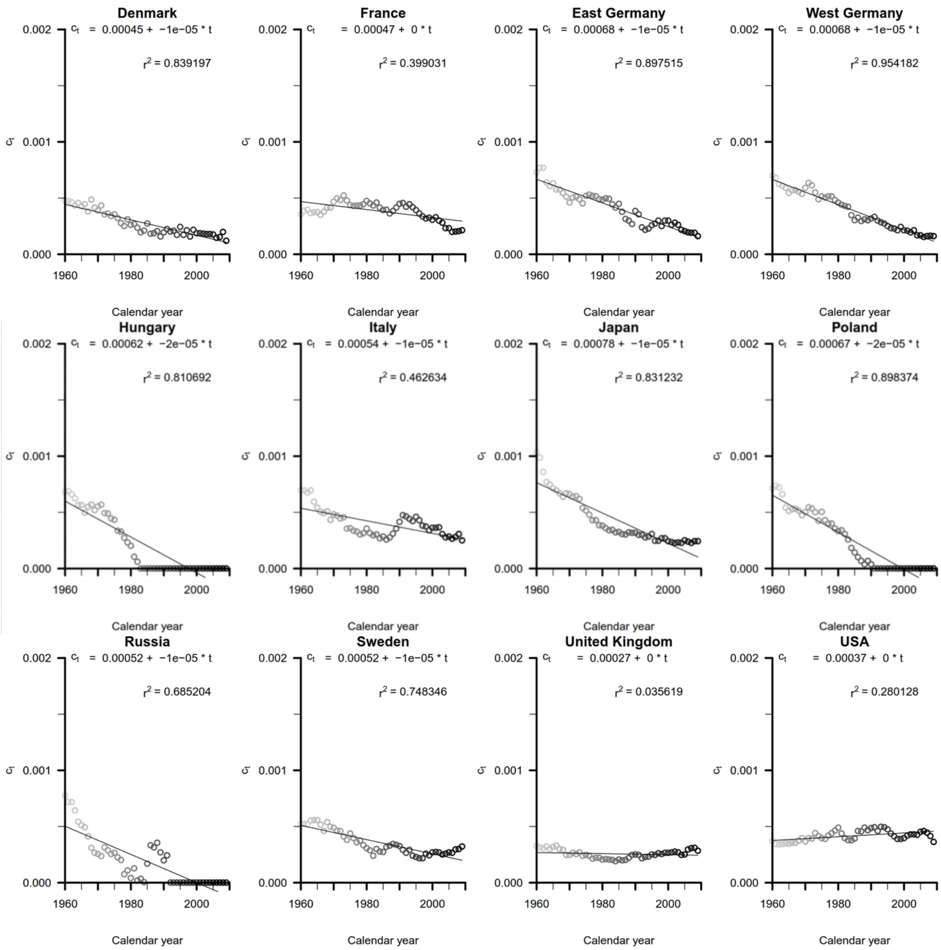

Figure A-17: Annual estimates for β_2 of the Siler model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

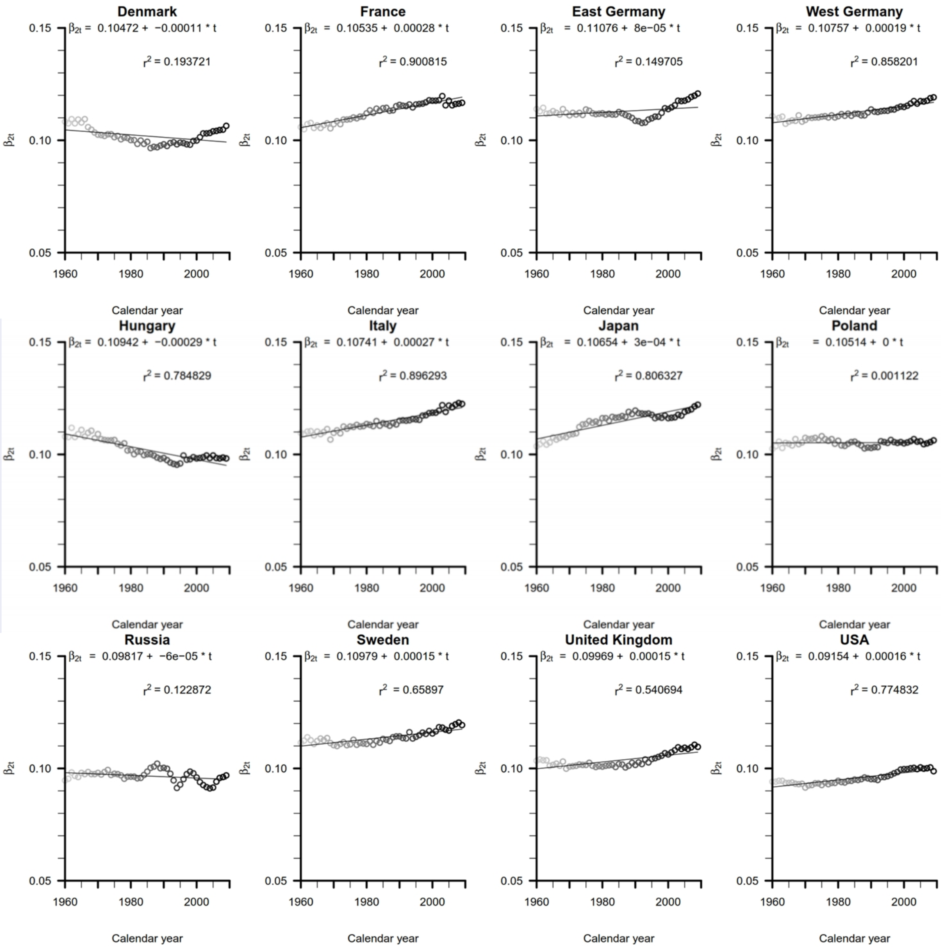

Figure A-18: Annual estimates for β_2 of the Siler model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

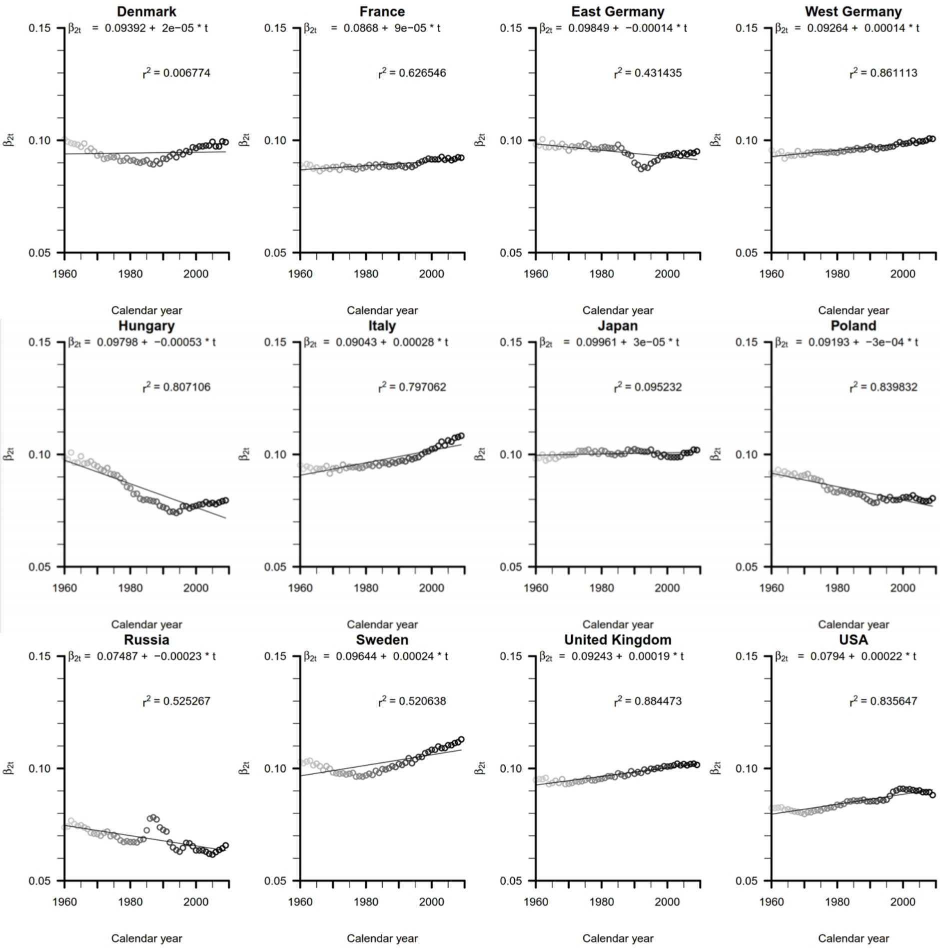

Figure A-19: Annual estimates for M of the Siler model, which is fitted to female mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)

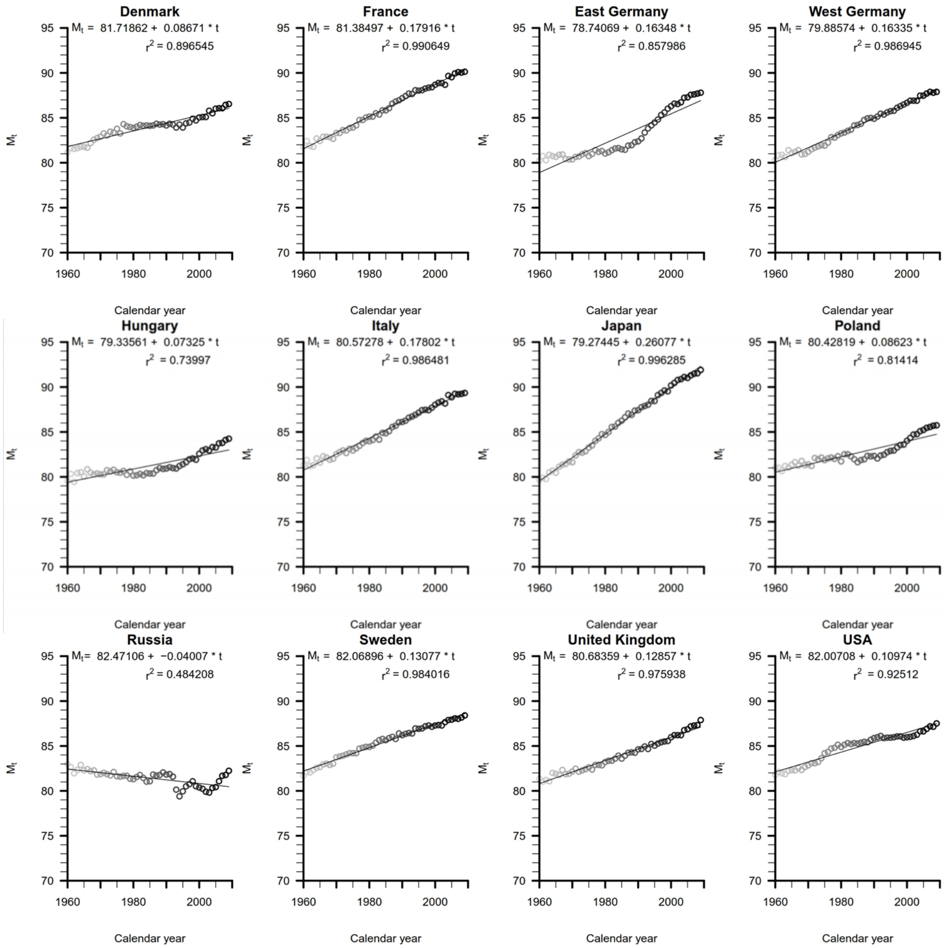

Figure A-20: Annual estimates for M of the Siler model, which is fitted to male mortality for the calendar years t, 1960 (light gray) to 2009 (black), along with a linear regression (solid line), for 12 countries of the Human Mortality Database (2015)