More general set of conditions producing a Taylor's Law with an exact slope of 2

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Dear Editor,

I would like to congratulate Joel E. Cohen, Christina Bohk-Ewald and Roland Rau (2018) on their paper "Gompertz, Makeham, and Siler models explain Taylor's law in human mortality data" published in *Demographic Research* on 01 March 2018.

One of the paper's findings, stated in the abstract, is that "the Gompertz model predicts Taylor's Law (TL) with a slope of exactly 2 if the modal age at death increases linearly with time and the parameter that specifies the growth rate of mortality with age is constant in time." I would like to propose here a more general set of conditions producing TL with a slope of exactly 2, of which Cohen et al.'s conditions are a special case:

Proposition:
When age-specific mortality rates change at a rate that is constant over time and identical for all ages, then Taylor's Law applies with an exact slope of 2. (No assumption about the age pattern of mortality.)

Proof:
When age-specific death rates change over time at a constant rate, the sequence of death rates over year t=0, 1, 2, 3, … for a given age *x* follows a geometric sequence: $\mu_{x,t} = \mu_{x,0}\, q^t$, where $\mu_{x,t}$ is the mortality rate for age *x* in year *t*, and *q* is the annual growth factor ("common ratio").

Using properties of geometric sequences, the *mean* of the sequence $\mu_{x,0},\ \mu_{x,1}, \mu_{x,2}, \ldots, \mu_{x,T}$ is equal to:

$$mean = \frac{\mu_{x,0}}{T+1}\ \frac{(1-q^{T+1})}{(1-q)} = \ \mu_{x,0} \cdot\ A$$

where $A = \frac{1}{T+1}\ \frac{(1-q^{T+1})}{(1-q)}$ A.

The *variance* of the sequence $\mu_{x,0},\ \mu_{x,1}, \mu_{x,2}, \ldots, \mu_{x,T}$ is equal to:

$$variance = \frac{{\mu_{x,0}}^2}{T+1}\ \frac{\left(1-q^{2(T+1)}\right)}{(1-q^2)} - \frac{{\mu_{x,0}}^2}{(T+1)^2}\ \frac{\left(1-q^{(T+1)}\right)^2}{(1-q)^2} = {\mu_{x,0}}^2 \cdot\ B$$

where $B = \frac{1}{T+1}\ \frac{\left(1-q^{2(T+1)}\right)}{(1-q^2)} - \frac{1}{(T+1)^2}\ \frac{\left(1-q^{(T+1)}\right)^2}{(1-q)^2}$.

Take the mortality rate for any two different ages *x* and *y* at baseline (t=0), and assume that they change over time at the same rate (i.e., same growth factor q):

Baseline rate	Mean	Variance
$\mu_{x,0}$	$\text{mean}_x = \mu_{x,0} \cdot A$	$\text{var}_x = {\mu_{x,0}}^2 \cdot B$
$\mu_{y,0}$	$\text{mean}_y = \mu_{y,0} \cdot A$	$\text{var}_y = {\mu_{y,0}}^2 \cdot B$

Calculating the change in ln(var) vs. ln(mean) between these two values produces:

$$\ln(\text{var}_y) - \ln(\text{var}_x) = 2\,[\ln(\text{mean}_y) - \ln(\text{mean}_x)]$$

Thus Taylor's Law applies with an exact slope of 2.

In their paper, Cohen et al. also use properties of geometric sequences to demonstrate their finding (Appendix 1). Their set of conditions (Gompertz model with constant β over time and linear increase with time in the modal age at death) also produces a TL with a slope of 2 because when these conditions are met, age-specific death rates all decline over time at the same rate. But as the above proof shows, this is a special case of a more general scenario of mortality change in which no assumption about the age pattern of mortality is required.

The above proof does not explain why observed mortality obeys TL with a slope generally less than 2, an observation that occupies a large portion of Cohen et al.' paper. However, the above development suggests that departures from a slope of 2 may have more to do with the fact that in reality rates of decline over time in age-specific death rates vary by age and time than with underlying age patterns of mortality.

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Reference

Cohen, J.E., C. Bohk-Ewald and R. Rau (2018). Gompertz, Makeham, and Siler models explain Taylor's law in human mortality data. *Demographic Research* 38(29), 773–842.