Dear Editor,

Among the interesting results in Cohen, Bohk-Ewald, and Rau's paper is the proof that Taylor's Law applies with an exact slope of 2 for any Gompertz mortality schedule when mortality changes at a constant multiplicative rate that is identical at all ages. Using Michel Guillot's notation, in which the (positive) multiplier is called q, the time series of mortality rates at any age x in the original proof can be written as

$$\left[\begin{array}{ccccc} \mu_{x,0} & \mu_{x,1} & \mu_{x,2} & \cdots & \mu_{x,T} \end{array} \right] = \mu_x^{Gompertz} \cdot \left[\begin{array}{ccccc} 1 & q & q^2 & \cdots & q^T \end{array} \right] \ \forall x$$

In his response letter, Prof. Guillot shows that the Gompertz assumption is unnecessary: with a constant multiplier q the "Taylor's Law" (TL) slope is exactly 2 for *any* initial age pattern of mortality $\{\mu_x\}$.

Here I point out a further generalization: although the multipliers for mortality must be identical at all ages in each time period, they need not be constant over time.

Proof Suppose, like Guillot, that there is an arbitrary initial mortality schedule $\{\mu_x\}$. Suppose also that there is a sequence of (all positive, but possibly different) multipliers $q_1, q_2, \ldots q_T$, so that the time series of mortality rates at age x is

$$\left[\begin{array}{ccccc} \mu_{x,0} & \mu_{x,1} & \mu_{x,2} & \cdots & \mu_{x,T} \end{array} \right] = \mu_x \cdot \left[\begin{array}{ccccc} 1 & q_1 & (q_1 q_2) & \cdots & (q_1 q_2 \cdots q_T) \end{array} \right] \ \forall x$$

Note that in this formulation mortality no longer has to be strictly increasing or decreasing over time. The level can fluctuate up and down, as long as all age-specific mortality rates move together.

In this generalized version the mean mortality rate over the time series at age x is

$$mean_x = \mu_x \cdot \frac{1 + q_1 + (q_1 q_2) + \cdots + (q_1 q_2 \cdots q_T)}{T+1} = \mu_x \cdot C_1 \qquad \forall x$$

where C_1 depends on the q-sequence, but is identical for all x. Analogously, the mean of the squared mortality rates over time at age x is

$$meansq_x = \mu_x^2 \cdot \frac{1 + q_1^2 + (q_1 q_2)^2 + \cdots + (q_1 q_2 \cdots q_T)^2}{T+1} = \mu_x^2 \cdot C_2 \qquad \forall x$$

where C_2 depends on the q-sequence, but is identical for all x. The relationship between the intertemporal mean and variance at each age is then

$$\begin{aligned} variance_x &= \mu_x^2 C_2 - (\mu_x C_1)^2 \\ &= \mu_x^2 \cdot C_1^2 \left(\frac{C_2}{C_1^2} - 1 \right) \\ &= (mean_x)^2 \cdot (\text{constant}) \end{aligned}$$

Taylor's Law therefore applies with a slope of exactly 2, because

$$\ln(variance_x) = \ln(constant) + 2 \ln(mean_x)$$

Implication In his letter, Prof. Guillot points to different rates of mortality change over ages and times as the likely cause of TL slopes less than 2 in empirical data. The generalization here shows that *purely temporal* variations in the rate of mortality change cannot explain TL slopes less than 2. The source must lie in the changing shapes of mortality schedules (i.e., differential change by age).

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Reference

JE Cohen, C Bohk-Ewald, and R Rau (2018). Gompertz, Makeham, and Siler models explain Taylor's Law in human mortality data. *Demographic Research* 38(29):773-842.