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Why does Taylor's law in human mortality data have slope less than 2, contrary to the Gompertz model?

Response by Joel E. Cohen, Christina Bohk-Ewald, Roland Rau to comments by Guillot and Schmertmann on:

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The central theoretical result of Cohen, Bohk-Ewald and Rau (2018; hereafter CBR) is the theorem in Appendix 1. It states: The Gompertz mortality model with modal age at death increasing linearly in time obeys a cross-age-scenario of Taylor's law (TL) exactly with slope $b = 2$. A cross-age-scenario of TL is a temporal TL in which the mean and variance of age-specific rates over time are calculated separately for each age group. We are delighted that our paper has stimulated Guillot and Schmertmann to discover illuminating generalizations.

Guillot teaches us that any initial age distribution (not only Gompertz') of age-specific mortalities such that every age's mortality rate changes geometrically by the same factor over time and at every age will obey TL exactly with slope $b = 2$.

Schmertmann teaches us that any time series of age-independent non-zero factors of change in age-specific mortality leads to TL exactly with slope $b = 2$, even if the factors change in time, as long as the same factors apply to changes at every age. We thank Guillot and Schmertmann for their valuable additions to theory.

CBR's central empirical result confirmed our earlier finding (Bohk et al. 2015) that observed mortality obeys TL with a slope generally (but not in every case) less than 2. So some assumption of the above mathematically correct theory is empirically wrong. According to CBR's empirical estimates, the two parameters of the Gompertz model, the modal age at death and the growth rate of mortality with age, both increased approximately linearly from year to year. The resulting Gompertz model was too complicated for CBR to extract much analytical insight (CBR, p. 799, Case 2).

Here we propose a simplified model to identify conditions under which mortality rates obey a cross-age-scenario of TL with slope $b < 2$ or $b > 2$. To summarize our main result in advance, we assume two age groups, young and old. We assume the young age group has lower average mortality over time than the old. We assume each age group's mortality declines geometrically at a rate that depends on the age group. We show that if mortality falls faster (over time) for the old than for the young, then $b > 2$, while if mortality falls faster (over time) for the young than for the old, then $b < 2$. These conclusions raise further empirical questions, which we begin to address after proving our main new theoretical result.

Now, the details. Generally, we follow the notation of CBR, except that, following Guillot and Schmertmann, we here let the index of time run from $t = 0$ to $t = T$ instead of from 1 to T as in CBR. We assume $0 < T < \infty$.

By way of background, the temporal mean of mortality μ at age x is defined by $E(\mu_x) := \frac{1}{T+1}\sum_{t=0}^{T}\mu_{x,t}$ (CBR (5)) and the temporal variance at age x is defined by $Var(\mu_x) := \frac{1}{T+1}\sum_{t=0}^{T}\left(\mu_{x,t} - E(\mu_x)\right)^2 = \frac{1}{T+1}\sum_{t=0}^{T}\mu_{x,t}^2 - \left(E(\mu_x)\right)^2$ (CBR (6)). TL asserts that $log_{10}Var(\mu_x) = a + b \cdot log_{10}E(\mu_x)$ (CBR (4)), or in the equivalent power-law form, $Var(\mu_x) = 10^a\left(E(\mu_x)\right)^b$. If TL holds, then $Var(\mu_x)/\left(E(\mu_x)\right)^2 = 10^a\left(E(\mu_x)\right)^{b-2} = \frac{1}{T+1}\sum_{t=0}^{T}\mu_{x,t}^2/\left(E(\mu_x)\right)^2 - 1$. Define the moment ratio as $R_x := \frac{1}{T+1}\sum_{t=0}^{T}\mu_{x,t}^2/\left(E(\mu_x)\right)^2$. Thus $Var(\mu_x)/\left(E(\mu_x)\right)^2 = R_x - 1$. Then, when TL holds,

$$R_x = 1 + 10^a\left(E(\mu_x)\right)^{b-2}.$$

From this expression, it is obvious that, when TL holds, R_x does not change with increasing mean mortality $E(\mu_x)$ if and only if $b = 2$, and in this case (only), R_x is unaffected by $\mu_{x,0}$. When TL holds, R_x increases with increasing mean mortality $E(\mu_x)$ if and only if $b > 2$, and R_x decreases with increasing mean mortality $E(\mu_x)$ if and only if $b < 2$. When either $b > 2$ or $b < 2$, R_x depends on $\mu_{x,0}$. We focus on the moment ratio R_x because it is simpler to analyze mathematically than the squared coefficient of variation $Var(\mu_x)/\big(E(\mu_x)\big)^2 = R_x - 1$, but it provides equivalent information about the slope b of TL.

Suppose mortality rates in each age group x decline geometrically by an age-specific factor q_x according to

$$\mu_{x,t} = \mu_{x,0} q_x^t, \quad ,0 < q_x < 1, \quad t = 0, 1, \dots, T.$$

For each age group x, the temporal mean (averaged over time) is (following Guillot)

$$E(\mu_x) = \mu_{x,0} A_x, \qquad A_x \coloneqq \frac{1}{T+1}(1 + q_x + \cdots + q_x^T) = \frac{1}{T+1}\frac{1 - q_x^{T+1}}{1 - q_x}.$$

Figure 1(a) plots $A_x = E(\mu_x)/\mu_{x,0}$ for $0 < q_x < 1$ and selected values of T. The temporal mean squared mortality (averaged over time) is (again following Guillot)

$$\frac{1}{T+1}\sum_{t=0}^{T} \mu_{x,t}^2 = \mu_{x,0}^2 C_x, \quad C_x \coloneqq \frac{1}{T+1}(1 + q_x^2 + \cdots + q_x^{2T}) = \frac{1}{T+1}\frac{1 - q_x^{2(T+1)}}{1 - q_x^2}.$$

Figure 1(b) plots $Var(\mu_x)/\mu_{x,0}^2 = (C_x - A_x^2)$ for $0 < q_x < 1$ and selected values of T. The moment ratio at age x depends on q_x according to

$$R_x = \frac{\mu_{x,0}^2 C_x}{\left(\mu_{x,0} A_x\right)^2} = \frac{\left[\frac{1}{T+1}\frac{1 - q_x^{2(T+1)}}{1 - q_x^2}\right]}{\left[\frac{1}{T+1}\frac{1 - q_x^{T+1}}{1 - q_x}\right]^2} = (T+1)\frac{1 - q_x^{2(T+1)}}{1 - q_x^2}\left[\frac{1 - q_x}{1 - q_x^{T+1}}\right]^2$$

$$= (T+1)\frac{1 - q_x^{2(T+1)}}{(1 - q_x^{T+1})^2}\left[\frac{(1 - q_x)^2}{1 - q_x^2}\right] = \left\{(T+1)\frac{1 + q_x^{T+1}}{1 - q_x^{T+1}}\right\}\left[\frac{1 - q_x}{1 + q_x}\right].$$

Figure 1(c) illustrates the decrease in R_x as a function of increasing q_x for finite values of T.

How does R_x behave with increasing q_x when T is large? The factor in curly braces on the right depends on T but the factor in square brackets does not. As $T \to \infty$, $q_x^{T+1} \to 0$ so $(T+1)\frac{1+q_x^{T+1}}{1-q_x^{T+1}}/(T+1) \to 1$, i.e., $(T+1)\frac{1+q_x^{T+1}}{1-q_x^{T+1}} \sim T+1$ in the conventional notation '~' for asymptotic approximation. As q_x increases from 0 to 1, $1 + q_x$ increases from 1 to 2, and $1 - q_x$ decreases from 1 to 0, so the ratio in square brackets $(1 - q_x)/(1 + q_x)$ decreases monotonically from 1 in the limit as $q \to 0$ to 0 in the limit as $q \to 1$. Therefore, for fixed large T, R_x decreases monotonically as q_x increases from 0 to 1. Explicitly, by elementary calculus and algebraic simplification, we find that

$$\frac{dR_x}{dq_x} = 2(T+1)\frac{\left(q_x^T - q_x^{T+2} + q_x^{2(T+1)} + Tq_x^T - Tq_x^{T+2} - 1\right)}{(1 - q_x^{T+1})^2(1 + q_x)^2}.$$

In the numerator of the fraction on the right, every term except the last, -1, goes to 0 as $T \to \infty$, and the denominator is always positive. So for increasing T the derivative is asymptotically negative and R_x asymptotically decreases monotonically as a function of increasing q_x.

Suppose we have only 2 age groups, the young (group 1) with mortality $\mu_{1,0}$ in year 0 and mortality change factor q_1; and the old (group 2) with mortality $\mu_{2,0} > \mu_{1,0}$ in year 0 and mortality change factor q_2.

We seek to find the slope b of TL as a function of the moment ratios in young and old. From $R_x = 1 + 10^a\big(E(\mu_x)\big)^{b-2}$, we subtract 1 from each side, then divide the equation for old, $x = 2$, by the equation for young, $x = 1$, and take logarithms, to find

$$\log_{10}\frac{R_2-1}{R_1-1}=(b-2)\log_{10}\frac{E(\mu_2)}{E(\mu_1)},$$

$$b=2+\frac{\log_{10}\frac{R_2-1}{R_1-1}}{\log_{10}\frac{E(\mu_2)}{E(\mu_1)}}.$$

Assume the temporal mean mortality of the old exceeds that of the young, i.e., $E(\mu_2) > E(\mu_1)$. It follows that $\log_{10}(E(\mu_2)/E(\mu_1)) > 0$. So whether the slope of TL satisfies $b > 2$ or $b < 2$ is determined by whether the numerator on the right is positive or negative, i.e., whether $R_2 > R_1$ or vice versa. We consider 2 cases.

Case 1. Suppose that mortality falls faster (over time) for the old than for the young, i.e., $0 < q_2 < q_1 < 1$. Then $R_2 > R_1$ and, by the above equation, $b > 2$.

Case 2. Suppose that mortality falls faster (over time) for the young than for the old, i.e., $0 < q_1 < q_2 < 1$. Then $R_2 < R_1$ and, by the above equation, $b < 2$.

Figure 1(d) illustrates both cases, with the additional assumpton that $E(\mu_2) = 10 \times E(\mu_1)$ so that $\log_{10}(E(\mu_2)/E(\mu_1)) = 1$.

This extremely simplified model, with only two age groups and mortality declining geometrically over time at a different rate in each age group, suggests hypotheses that can and should be tested empirically. How accurate is the model of geometrically declining mortality for different age groups? If that model is supported (even approximately), how do the factors of change in mortality q_x compare for different age groups? If that model of geometric change is not supported, then how do the cumulative products of the factors of change in mortality at each age compare for different age groups?

Rau et al. (2018) analyzed annual rates of improvement in smoothed estimates of mortality rates from 1950 to 2014 in 19 countries, including the 12 countries analyzed by CBR. The assumption above of geometrically declining mortality rates (equivalent to a constant rate of mortality improvement) is clearly far from the facts in their Chapter 6. Though rates of mortality improvement varied over time, their analyses make it easy to compare factors of change in mortality for different age groups. In many cases, such as women in France (Rau et al. 2018, p. 53, Fig. 6.9) and Italy (Rau et al. 2018, p. 57, Fig. 6.13), in many years between 1950 and 2014, mortality fell faster at younger ages than at older ages. For women in France and Italy and in other cases, CBR estimated $b < 2$. So there is at least qualitative compatibility between the assumption of Case 2 above and the estimate that $b < 2$. Exact necessary and sufficient conditions for the slope of TL to be below or above 2 in a realistic age-structured model remain to be determined. The cartoon model we present here at least offers some insight and raises clear questions.

We thank Guillot and Schmertmann for inspiring these further reflections on the origin, parameters, and interpretation of Taylor's law in human mortality data.

References

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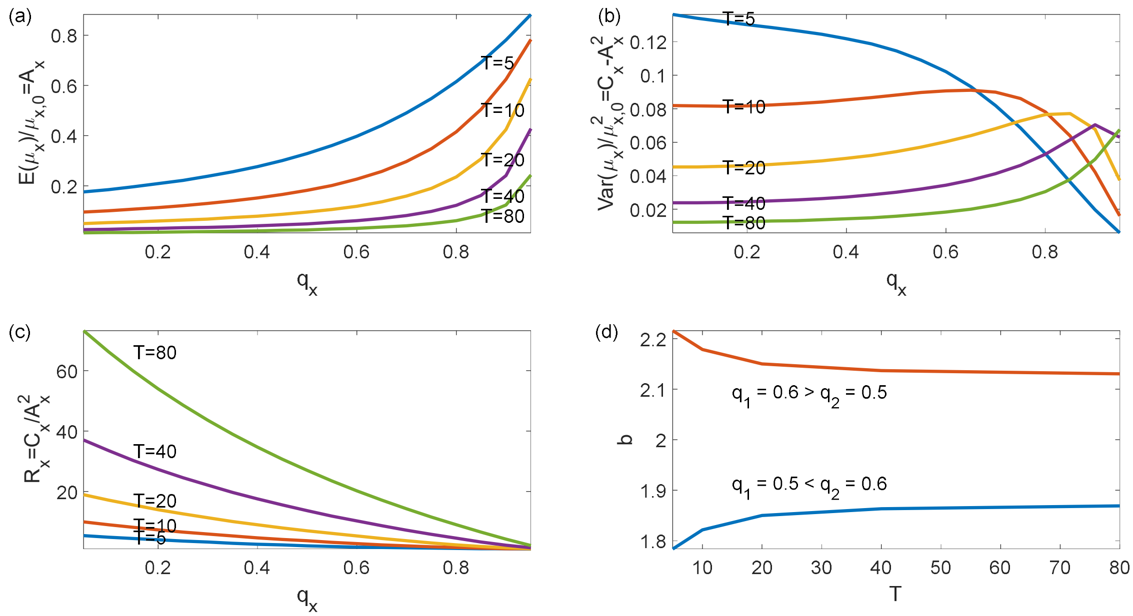

Figure 1. (a) $A_x = E(\mu_x)/\mu_{x,0}$ as a function of the factor q_x of decline in age-specific mortality for $0 < q_x < 1$ and selected time horizons T. (b) $Var(\mu_x)/\mu_{x,0}^2 = (C_x - A_x^2)$ for $0 < q_x < 1$ and selected values of T. (c) Moment ratio $R_x = C_x/A_x^2$ for $0 < q_x < 1$ and selected values of T. (d) Taylor's law slope b for selected time horizons T in Case 1, $q_1 = 0.6 > q_2 = 0.5$ with $b > 2$, and Case 2, $q_1 = 0.5 < q_2 = 0.6$ with $b < 2$. Text gives definitions of notation.