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Formal Relationship 27

The impact of proportional changes in age-specific mortality on life expectancy when mortality is a log-linear function of age

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Abstract

BACKGROUND
Demographers and epidemiologists have investigated the dependence of life expectancy on proportional changes in the age-specific mortality.

OBJECTIVE
To develop an approach that allows estimation of change in life expectancy from proportional changes in age-specific mortality and to identify aspects of the death rate that influence the accuracy of the estimation.

RESULTS
We obtain an exact expression for the first derivative of the life expectancy with respect to the proportional change in age-specific mortality when the age-specific death rate is log-linear in age. We use the result to establish bounds for the change in life expectancy following a proportional change of the mortality. The result shows that the change in life expectancy is approximately linear in the logarithm of the proportional change of the mortality. In populations with low infant mortality, the slope of this linear relationship is essentially equal to minus the inverse of the slope of the log-linear age dependence of the death rate.

CONTRIBUTION
In a wide range of mortality scenarios, the relationship between change in life expectancy and the logarithm of the proportional change of the mortality allows accurate approximation of a difference in life expectancy from a ratio of death rates.

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1. Relationship

Assume that lifetime from a given age, a, follows a Gompertz distribution whose age-specific death rate is defined as $m(x) = \alpha e^{\beta(x-a)}$ at age $x \geq a$ and let $\kappa = \alpha\beta^{-1}$.

Consider a proportional change of age-specific mortality by a constant c and let $e(a;c)$ denote life expectancy at age a when the age-specific death rate is $m_c(x) = cm(x)$.

The change in life expectancy introduced by the proportional change c in age-specific mortality then satisfies the following inequalities:

$$\beta^{-1}\log(c)h(c\kappa) < e(a;1) - e(a;c) < \beta^{-1}\log(c)h(\kappa), \tag{1}$$

with $h(x) = 1 - xe^x E_1(x)$, where $E_1(x) = \int_x^\infty \frac{e^{-z}}{z}dz$ is the exponential integral.

2. Proof

Consider life expectancy as a function of $b = \log(c)$. The mean value theorem then gives

$$e(a;c) - e(a;1) = (b-0)D_b e(a;\exp(b^*)), \tag{2}$$

where the first derivative with respect to b is computed for some $b^* \varepsilon [0,b]$. To establish the inequalities, we show that life expectancy $e(a;c)$ is a decreasing, convex function of b such that the first derivative at b^* is bounded between the values of the first derivative at the boundary of the interval.

It is well-known (see, e.g., Broadbent 1958) that $e(a;1) = \beta^{-1}e^{\kappa}E_1(\kappa)$. If the age-specific mortality is changed by a factor of c for ages $x \geq a$, lifetime follows a Gompertz distribution with parameters $c\alpha, \beta$. Therefore,

$$e(a;c) = \beta^{-1}\exp(c\kappa)E_1(c\kappa). \tag{3}$$

Straightforward calculations give

$$D_b e(a;c) = -\beta^{-1}[1 - c\kappa e^{c\kappa}E_1(c\kappa)] = -\beta^{-1}h(c\kappa), \tag{4}$$

and

$$D_b^2 e(a;c) = \beta^{-1}[c\kappa e^{c\kappa}E_1(c\kappa)(c\kappa+1) - c\kappa].$$

From Temme (2010) we have (see formula 6.8.2)

$$\frac{x}{x+1} < xe^x E_1(x) < \frac{x+1}{x+2},$$

so $(x+2)^{-1} < h(x) < (x+1)^{-1}$ showing that $D_b e(a;c) < 0$. Moreover, the second derivative satisfies the inequalities

$$0 < D_b^2 e(a;c) < \beta^{-1}(c\kappa+2)^{-1},$$

and the first derivative is therefore an increasing function of b, i.e.,

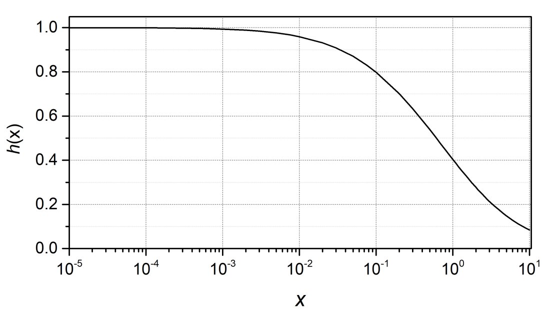
$$D_b e(a;1) < D_b e(a;\exp(b^*)) < D_b e(a;c).$$

The result in (1) then follows by applying these inequalities to (2) and rearranging terms.

2.1 Comments

For small values of $c\kappa$ and κ, the inequalities in (1) are very tight. The derivation above shows that the correction factor *h(x)* is a decreasing function of *x*. Moreover, $\lim_{x\to 0} h(x) = 1$ and, using the power series representation of $E_1(x)$ – see Temme (2010), formula 6.6.2 – it follows that *h(x)* behaves like $1 + x\log(x)$ as $x \to 0$. In particular, if $c\kappa$ and κ are both less than 0.0017, then the ratio of the lower bound to the upper bound in (1) is between 0.99 and 1. Consequently, the change in life expectancy is essentially proportional to $\log(c)$ with $-\beta^{-1}$ as the constant of proportionality. Figure 1 shows the behavior of *h(x)* in the range from 0+ to 10.

Figure 1: The correction factor $h(x) = 1 - xe^x E_1(x)$ **for** $x < 10$



Note: Note the logarithmic scale on the x-axis.

3. Robustness of the result

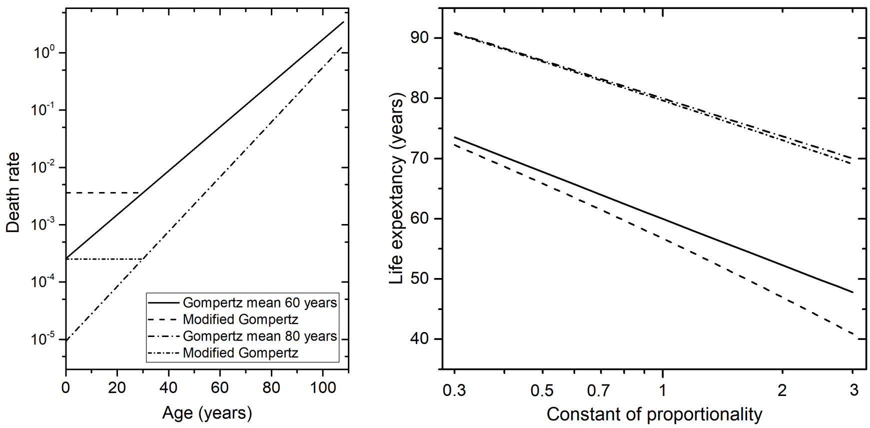
The distribution of lifetimes in human populations differs from that predicted by a Gompertz distribution because of a higher mortality in childhood and for young adults. Moreover, the slope of the log-linear age relation of the mortality rate is reduced for very old ages. The sensitivity of the main result to such deviations is therefore of some interest. To this end we consider the impact of two mathematically tractable modifications of the death rate.

A simple modification that allows higher mortality at young ages may be obtained by assuming a constant death rate $m(x) = \alpha$ for ages $x \leq A$. For ages $x > A$, the log-linear age dependence takes over such that $m(x) = \alpha e^{\beta(x-A)}$. Note that this modification sets the death rate for ages younger than A to the death rate at age A for the Gompertz model. Life expectancy at birth in this distribution becomes

$$(5) \qquad e_A(0) = \alpha^{-1}(1 - e^{-\alpha A}) + e^{-\alpha A}\beta^{-1}e^{\kappa}E_1(\kappa) = \alpha^{-1}(1 - e^{-\alpha A}) + e^{-\alpha A}e(A;1),$$

where $\kappa = \alpha\beta^{-1}$ and $e(A;1)$ are defined in (3). Note that in this model, $e^{-\alpha A}$ is the probability of surviving until age A. Replacing α by $c\alpha$ introduces a proportional change of the mortality, and the change in life expectancy as a function of $b = \log(c)$ may then be studied. Figure 2 shows the results for two scenarios in which a Gompertz distribution has been modified to have a constant death rate for ages below 30. The solid curve in the right panel shows the relation for a Gompertz distribution with life expectancy 60 years and a probability of surviving until age 30 years of $p(30) = 0.963$. A similar modification is shown for a Gompertz distribution with life expectancy 80 years and $p(30) = 0.998$. The modification of the Gompertz distribution with mean 80 years has almost no impact on the relationship, but for the other scenario the effect is clearly visible, in particular for $c > 1$.

Figure 2: Death rates and life expectancy as a function of a proportional change of the mortality



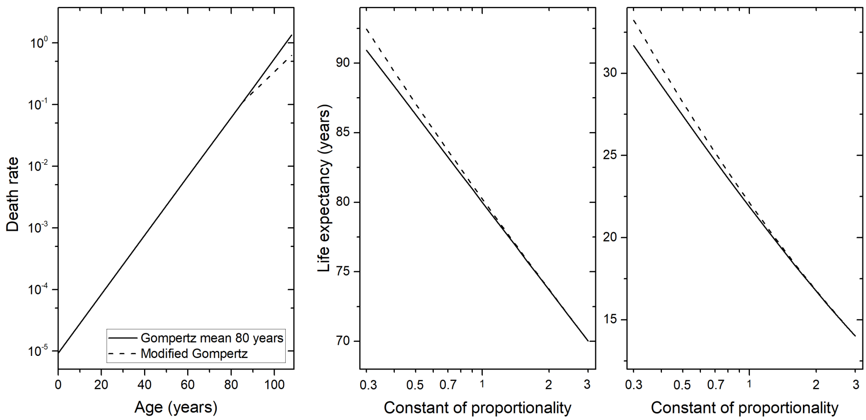
Note: The left panel shows the death rate of a Gompertz distribution with mean 60 years, $(\alpha,\beta) = (2.562\times 10^{-4}, 0.088)$, a modification of this distribution with $m(x) = 3.591\times 10^{-3}$ for $x < 30$, a Gompertz distribution with mean 80 years, $(\alpha,\beta) = (9.317\times 10^{-6}, 0.11)$, and a modification of this distribution with $m(x) = 2.526\times 10^{-4}$ for $x < 30$. For these four distributions, the right panel shows the relation between $\log(c)$ and the change in life expectancy at birth for $0.3 < c < 3$ (equations (3) and (5)).

The impact of a reduced log-linear slope at old ages may conveniently be studied by assuming that $m(x) = \alpha e^{\beta(x-a)}$ for $a \le x < B$ and $m(x) = \alpha_B e^{\beta_1(x-B)}$ for $x > B$, where $\alpha_B = \alpha e^{\beta(B-a)}$. In this distribution, the life expectancy at age a is given by

$$e_B(a) = \beta^{-1}e^{\kappa}E_1(\kappa) + p(B \mid a)(\beta_1^{-1}e^{\kappa_2}E_1(\kappa_2) - \beta^{-1}e^{\kappa_1}E_1(\kappa_1)), \tag{6}$$

where $\kappa = \alpha\beta^{-1}, \kappa_1 = \alpha_B\beta^{-1}, \kappa_2 = \alpha_B\beta_1^{-1}$ and where $p(B \mid a) = \exp(-\alpha\beta^{-1}(e^{\beta(B-a)} - 1))$ is the probability of surviving to age B given survival until age a in a Gompertz distribution with parameters α and β. The first term is the life expectancy in this Gompertz distribution, so the modification introduced by the second term is obviously small when the survival probability $p(B \mid a)$ is small. Again, a proportional change of the mortality is introduced by replacing α with $c\alpha$.

Figure 3: Death rates and life expectancy as a function of a proportional change of the mortality



Note: The left panel shows the death rate of a Gompertz distribution with mean 80, $(\alpha, \beta) = (9.317 \times 10^{-6}, 0.11)$, and of a modification for which the log-linear slope is reduced by 30% for ages above 85. For $0.3 < c < 3$, the center panel shows the life expectancy at birth as a function of $\log(c)$, and the right panel shows the life expectancy at age 60 years as a function of $\log(c)$, cf. equations (3) and (6).

Figure 3 illustrates the relation between $\log(c)$ and the life expectancy at birth (center panel), and at age 60 years (right panel), when there is a 30% decrease in the log-linear slope at ages above 85 years. The unmodified Gompertz distribution is identical to the distribution with mean 80 years used in Figure 2. In this distribution the probability of surviving until age 85 years is 0.378. The probability becomes smaller when c increases, and the modification therefore has little or no impact for $c > 1$. For $c < 1$, the slope in Figure 3 becomes steeper and, as c decreases, approaches the value of minus the inverse of the modified log-linear slope. This modification introduces some curvature.

While not exhaustive, the results of this sensitivity study suggest that the approximately linear relation between $\log(c)$ and life expectancy is relatively robust to the type of deviations from a Gompertz distribution observed in human populations.

4. Previous work on the relation between mortality rate and life expectancy

The relation between different demographic parameters has been studied for centuries. More than 250 years ago, Bernoulli and d'Alembert investigated the impact on survivorship and life expectancy by eliminating a cause of death (Karn 1931). More recently, changes in life expectancy due to small proportional changes in the total or cause-specific mortality have been described by Keyfitz (1977, 1985), Vaupel (1986), and Vaupel and Canudas Romo (2003). These and other results have been summarized by Wrycza and Baudisch (2012), who also present results on changes in life expectancy that are introduced by other types of perturbations in age-specific mortality.

Demographers are often interested in the dynamics of populations, and their focus is therefore on describing the impact of small changes on population parameters from one calendar year to the next. Epidemiologists have also studied the impact of changes in the age-specific mortality on life expectancy, but their purpose has been different. In epidemiology, both the life expectancy and the standardized mortality ratio (SMR) are used as mortality indices for population comparisons, and a relationship between them would allow a translation of results from one index to the other. Tsai, Hardy, and Wen (1992) and Lai, Hardy, and Tsai (1996) used regression methods to establish an empirical relation between SMR and life expectancy, whereas Haybittle (1998) applied a result for Gompertz distributions developed by Pollard (1991) to obtain an approximate linear dependency of life expectancy on the logarithm of the SMR. Our main result allows a study of the validity of their approximations.

Spiegelhalter (2016) presented a result closely related to our results. Assuming a log-linear age dependency of the death rate, he showed that the effect of a proportional change of the death rate by a constant c can be expressed as an adjustment of the current age by $\beta^{-1}\log(c)$ years, where β is the slope of the log-linear age dependence. He described this adjusted age as the "effective age," and advocated its use for risk communication. Thus, Spiegelhalter described the effect of the mortality change by moving the starting point of the remaining lifetime, whereas our approach involves changing the expected endpoint of the remaining lifetime.

5. An application

In a human population, let $p(x)$ denote the probability of surviving until age x. After a proportional change of the mortality by a factor of c, life expectancy at age a becomes

$$a(a;c) = \int_a^\infty p(x \mid a)^c dx, \tag{7}$$

where $p(x \mid a) = p(x)/p(a)$.

The results in the previous sections suggest that $e(a;c)$ is approximately linear in $\log(c)$. When we therefore introduce a proportional change to the mortality by a factor of c, the change in life expectancy at age a will to a close approximation be equal to

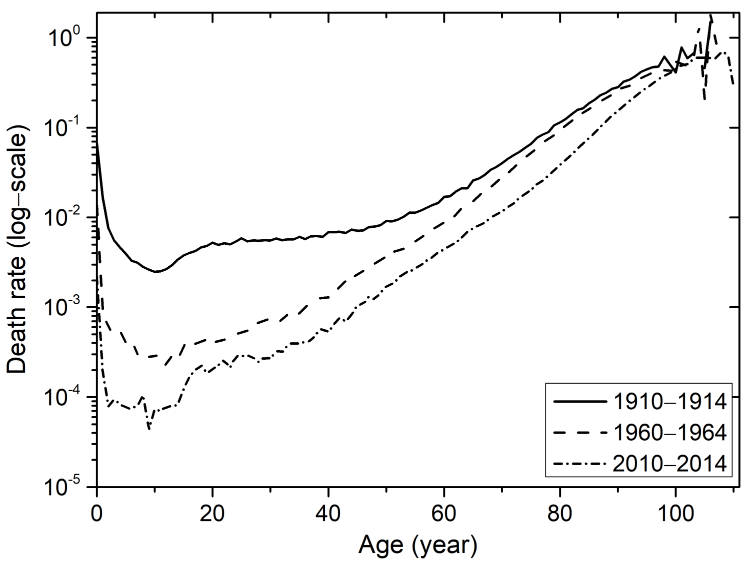
$$e(a;c) - e(a;1) \approx K \ln(c), \tag{8}$$

where K is the slope of the change. K is equal to the first derivative of the life expectancy with respect to $b = \log(c)$ computed at $b = 0$, i.e.,

(9) $$K = \int_a^\infty \log(p(x \mid a))p(c \mid a)dx.$$

To study the validity of the linear approximation (8) in a range of mortality scenarios, we consider data on Swedish women in three periods, 1910–1914, 1960–1964, and 2010–2014. From the Human Mortality Database (Human Mortality Database 2017), we obtained period life tables based on one-year age categories including data on number of deaths, exposure to risk, and mortality rates. The final age category was 110+ years. Figure 4 shows the death rates for the three periods as a function of age. (Note the log-scale on the y-axis.) For adult ages, the logarithm of the death rate is approximately linear in age.

Figure 4: Death rates for Swedish women in the periods 1910–1914, 1960–1964, and 2010–2014

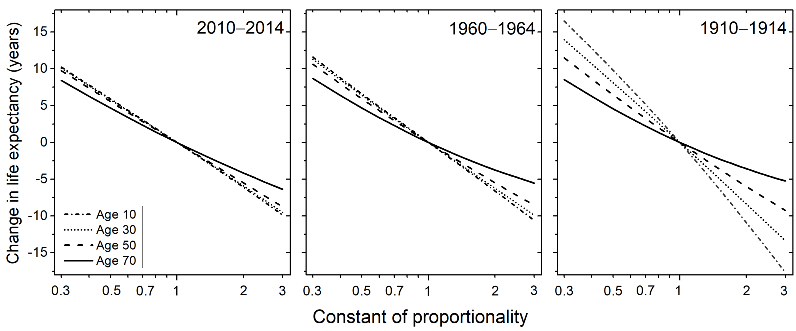


Source: Human Mortality Database 2017.

Figure 5 shows the change in life expectancy as a function of the logarithm of the proportional change of the mortality. For each period, the change in life expectancy is computed from the life table using formula (7) with age a = 10, 30, 50, and 70. The curves are remarkably linear, in particular for the periods 1960–1964 and 2010–2014. For these periods, the slope of the change in life expectancy at age 70 is less steep, and the curve is slightly convex. For the earliest period, the slopes differ more, and some curvature is notable for age 10 and age 70.

Two approaches were used to estimate the constant K, a direct life table calculation based on numerical integration of the right side of (9), and an approach relying on the relation $K = -h(\kappa)\beta^{-1}$ for a Gompertz distribution, cf. formula (4). The Gompertz model was fitted by a Poisson regression model with logarithmic link function, number of deaths as the dependent variable, and the logarithm of the exposure to risk as an offset. For each period, the analysis was carried out for ages a = 10, 30, 50, and 70. Stata version 14 was used for all analyses (StataCorp 2015).

Figure 5: Change in life expectancy at age *a*; *a* = 10, 30, 50, and 70 as a function of a proportional change of the mortality rate



Note: Change in life expectancy computed from equation (7). Note the log-scale on the x-axis.
Source: Swedish females 1910–1914, 1960–1964, and 2010–2014; Human Mortality Database 2017.

Table 1 summarizes the results on estimation of the slope K for the linear approximation (6). The estimates derived from the Gompertz model agree remarkably well with the slopes obtained by direct numerical integration, and also when computed for a = 10, where the fit of the Gompertz model is inadequate.

Table 1: Life expectancy at age *a*, and comparison of –*K* estimated from a Gompertz analysis and from a life table calculation

Period	Age *a*	Life expectancy	–*K*, Gompertz	–*K*, num. integration	Rel.Error
2010–2014	10	73.9	8.56	8.61	0.5%
	30	54.2	8.39	8.40	0.2%
	50	34.7	7.94	7.95	0.1%
	70	17.0	6.34	6.36	0.3%
1960–1964	10	66.8	9.43	9.44	0.2%
	30	47.3	9.03	9.04	0.1%
	50	28.4	8.18	8.20	0.3%
	70	12.0	5.97	6.04	1.2%
1910–1914	10	56.5	15.25	14.92	–2.2%
	30	40.7	11.82	11.88	0.5%
	50	24.9	8.92	8.97	0.7%
	70	10.6	5.79	5.86	1.3%

Note: In the Gompertz analysis. –*K* is estimated as $-h(\kappa)\beta^{-1}$.
Source: Swedish women 1910–1914, 1960–1964, and 2010–2014; Human Mortality Database 2017.

According to the Swedish life table for the period 2010–2014, life expectancy is 54.17 years for a woman at age 30. If the age-specific death rates are reduced by 50%, the life expectancy obtained by numerical integration of (7) becomes 59.98 years, and the linear approximation (8) leads to 60.00 years. For the three periods considered here, Table 2 compares the exact change in life expectancy at age a = 10, 30, 50, and 70, obtained with numerical integration and the approximate value derived from the linear approximation (8) when the age-specific death rate is reduced by 50% or increased by a factor of 2. The results in Table 2 show that the linear approximation (8) provides an accurate estimate of the change in life expectancy following a relatively large proportional change of the mortality, in particular when the life expectancy is computed from age 30 or age 50. As expected, the accuracy seems to be highest when infant mortality is small.

Overall, the results in Table 1 and 2 indicate that the linear approximation will provide a reasonable approximation in a wide range of mortality scenarios provided that the logarithm of the mortality rate increases proportionally with age for adult ages.

Table 2: Comparison of exact values and approximate values of the change in life expectancy at age *a* following a proportional change in death rates by a factor of 0.5 and a factor of 2

		Proportional change					
		c = 0.5			**c = 2**		
Period	**Age *a***	**Exact**	**Approx.**	**Rel.Error**	**Exact**	**Approx.**	**Rel.Error**
2010–2014	10	5.91	5.97	0.9%	–6.11	–5.97	–2.4%
	30	5.80	5.82	0.3%	–5.92	–5.82	–1.6%
	50	5.56	5.51	–0.9%	–5.51	–5.51	0.0%
	70	4.67	4.41	–5.6%	–4.16	–4.41	5.9%
1960–1964	10	6.60	6.55	–0.7%	–6.60	-6.55	–0.9%
	30	6.38	6.27	–1.8%	–6.23	–6.27	0.6%
	50	5.92	5.68	–4.0%	–5.49	–5.68	3.6%
	70	4.66	4.19	–10.1%	–3.74	–4.19	12.1%
1910–1914	10	9.81	10.34	5.3%	–10.90	–10.34	–5.2%
	30	8.10	8.23	1.7%	–8.38	–8.23	–1.8%
	50	6.44	6.22	–3.5%	–5.99	–6.22	3.9%
	70	4.57	4.06	–11.0%	–3.58	–4.06	13.5%

Note: Approximate values are based on equation (8).
Source: Mortality data on Swedish women 1910–1914, 1960–1964, and 2010–2014; Human Mortality Database 2017.

Proportional changes of the age-specific mortality of the order 2 or larger from one calendar year to the next are extremely unlikely in demographic applications. A relationship between the change in mortality and the change in life expectancy that is valid for a change of a few percentages will usually suffice. However, life expectancy and the standardized mortality ratio are also used as mortality indices for comparison of populations, and a relationship between them will be useful if it can be applied in a wide range of scenarios. In Skriver, Væth, and Støvring (2018), we explore to what extent the linear approximation described above may be used together with the standardized mortality ratio to estimate a difference in life expectancy between two populations using one of the populations as the standard population.

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