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Research Article

A probabilistic model for analyzing summary birth history data

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Katie Wilson¹

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Abstract

BACKGROUND

There is an increasing demand for high-quality subnational estimates of under-5 mortality. In low- and middle-income countries, where the burden of under-5 mortality is concentrated, vital registration is often lacking, and household surveys, which provide full birth history data, are often the most reliable source. Unfortunately, these data are spatially sparse so data are pulled from other sources to increase the available information. Summary birth histories represent a large fraction of the available data and provide numbers of births and deaths aggregated over time, along with the mother's age.

OBJECTIVE

Specialized methods are needed to leverage this information, and previously the Brass method and variants have been used. We wish to develop a model-based approach that can propagate errors and make the most efficient use of the data. Further, we strive to provide a method that does not have large computational overhead.

CONTRIBUTION

We describe a computationally efficient model-based approach that allows summary birth history and full birth history data to be combined into analyses of under-5 mortality in a natural way. The method is based on fertility and mortality models that allow smoothing over time and space, with the possibility for including relevant covariates associated with fertility and/or mortality. We first examine the behavior of the approach on simulated data before applying the model to household survey and census data from Malawi.

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1. Introduction

The United Nations' Sustainable Developmental Goals emphasize the importance of sub-national estimates of the under-5 mortality risk (U5MR). Subnational estimates of U5MR are an important measure of the health of a nation and are used to inform public policy. Unfortunately, countries where U5MR is highest often lack vital registration systems that track births and deaths. Instead, data are collected from household surveys and censuses. Many surveys, such as the Demographic and Health Surveys (DHS), provide full birth history (FBH) data, which consist of the dates of birth and death (if applicable) of the children of surveyed women. FBH data can be used to obtain time-varying subnational estimates of U5MR (in regions where enough FBH data is collected). See, for example, Li et al. (2019). Many censuses (and some surveys) collect summary birth history (SBH) data consisting of the number of births and deaths of surveyed women. SBH information is far easier to collect, hence its inclusion on the census in many low- and middle-income countries (LMIC). No temporal information on when the births and deaths occurred is provided in SBH data, however; therefore, incorporation of this data is more challenging. However, the SBH data benefit from the large sample size of the census. With both FBH and SBH data, other variables, such as the age of the woman, are also available. SBH data are more common; in a study of U5MR in Africa, 90% of the births were from SBH data (Golding et al. 2017).

The wish to exploit SBH data has long been desired, beginning with the famous approach introduced by Brass (1964). The Brass method, which we describe in detail in the next section, is essentially a deterministic approach, which has seen many variants since its introduction (Brass 1975; Coale and Trussell 1977; Feeney 1976; Hill, Zlotnik, and Trussell 1983; Sullivan 1972; Trussell 1975). A commonly used approach for obtaining standard errors for Brass estimates, which may be used to construct confidence intervals, is the jackknife (Pedersen and Liu 2012). Rajaratnam et al. (2010) refine the Brass method and use observed FBH data in place of model life tables. More recently, Burstein et al. (2018) describe a further extension of the approach that introduced a discrete survival model, with birth times assigned to SBH children. Verhulst (2016) compares various SBH models using simulated data. Hill et al. (2015) propose two alternatives, including the birth history imputation method, in which SBH women are matched to FBH women who are of approximately the same age and have the same numbers of both births and deaths. They also describe the cohort change method, which requires data from SBH surveys taken 1 to 2 years apart. They posit that the change in number of children who died and number of children born will largely be driven by U5MR. Thus they leverage these observed quantities in the SBH data to derive an estimate of U5MR. Unfortunately, the two methods provided very mixed results when applied to real data (Brady and Hill 2017).

Perhaps surprisingly, the above body of work for the analysis of SBH data does not

take a fully probabilistic approach, in which a model is specified at the child level, with uncertainty correctly propagated. By “fully probabilistic” we mean a model in which probability distributions are specified for both the birth and death times. Such a fully stochastic model allows FBH and SBH data to be combined in a coherent way, since they are both built on common underlying models for birth and death. The approach can be tuned to the context at hand; for example, spatial and temporal smoothers can be incorporated, along with indicators of urban/rural and other covariates. Further, bias terms can be included to allow for systematic differences between the FBH and SBH data that may exist in particular contexts. Inference follows in a straightforward, conventional fashion, since a regular statistical modeling framework is used.

In the context of analyzing FBH data, a modeling framework based on space-time modeling of discrete hazards is available and summarized in Wakefield et al. (2019). Previously, Wilson and Wakefield (2021) proposed a model-based approach in which the unknown birth and death times in SBH data are simulated and then combined with FBH data to obtain estimates of U5MR in a data augmentation framework. While this method allows for flexible inclusion of covariates and can accommodate bias in surveys, it is computationally demanding, since typically the birth and death times of millions of children must be repeatedly simulated. Unfortunately, prior to Wilson and Wakefield (2021), approaches to analyzing SBH data did not easily support a space-time modeling framework, since they were not based on providing probability models directly for the mortality hazards.

In this paper, we propose a computationally efficient approach using a Poisson approximation that removes the simulation step of Wilson and Wakefield (2021). By greatly easing the computational burden we can extend the modeling approach and incorporate more complex space-time models. We fit the model using the R package TMB (with the acronym standing for Template Model Builder), which allows one to quickly obtain empirical Bayes estimates along with other inferential summaries (Kristensen 2014; Kristensen et al. 2016). The paper is laid out as follows. We begin by describing the Brass method before detailing the proposed method. We next describe a simulation study, in which we study the performance of our method and then apply the method to data from Malawi. For the latter, FBH data from five household surveys is combined with census SBH data to give yearly U5MR estimates in 26 regions of Malawi; this is the first comprehensive analysis of child mortality data in Malawi that rigorously combines FBH and SBH data. The paper concludes with a discussion.

2. Methods

2.1 The Brass method

We describe how to implement the Trussell version of the Brass method, which uses the Coale-Demeny model life tables. This description is adapted from chapter 3 of Hill, Zlotnik, and Trussell (1983), using the same notation. We index the five-year age groups of women, 15–19, ..., 45–49, by $i = 1, \dots, 7$, and let $q(x)$ represent the probability that a child dies before age x . The basic idea is to equate the fractions of children who died to mothers of different ages to mortality probabilities $q(x)$ for an appropriate x . The method is described by the following steps:

- (1) Calculate the average parity per woman by age group i ,

$$P(i) = \frac{CEB(i)}{FP(i)},$$

where $CEB(i)$ is the number of children ever born to, and $FP(i)$ is the total number of women in, age group i .

- (2) Calculate the fraction of children who died to mothers in age group i , denoted $D(i)$, with

$$D(i) = \frac{CD(i)}{CEB(i)},$$

where $CD(i)$ is the number of children who died to mothers in age group i .

- (3) Next, convert the fraction of children who died to mothers in age group i to $q(x)$ for a particular x . Specifically, calculate the probability of dying by age x ,

$$q(x) = D(i) \left(a(i) + b(i) \frac{P(1)}{P(2)} + c(i) \frac{P(2)}{P(3)} \right),$$

where the coefficients $a(i)$, $b(i)$, and $c(i)$ are estimated from a simulation based on the Coale-Demeny model life tables and x is given in a table in the supplementary materials. For example, the proportions of child deaths in mothers in age group 15–19 maps to $q(1)$ and those in the 30–34 group to $q(5)$.

- (4) Given that mortality and fertility change over time, appropriate reference dates for

$q(x)$ are calculated as

$$t(i) = e(i) + f(i) \frac{P(1)}{P(2)} + g(i) \frac{P(2)}{P(3)},$$

where the coefficients $e(i)$, $f(i)$, and $g(i)$ are estimated from a simulation based on the Coale-Demeny model life tables. These coefficients are taken from Table 48 of Hill, Zlotnik, and Trussell (1983).

- (5) If U5MR is of interest, convert $q(x)$ to $q(5)$. This is done by identifying the mortality “level” (expectation of life) in a table that most closely corresponds to the observed $q(x)$ and then using the table to convert $q(x)$ to the desired index. There are different tables for each of the Coale-Demeny models. In practice, linear interpolation is used to interpolate between different levels in the life table. Suppose $q^e(x)$ is the estimated value of $q(x)$ obtained in step 3. In the table, find the level j such that

$$q^{j+1}(x) < q^e(x) < q^j(x).$$

Then,

$$q(5) = (1 - h)q^j(5) + hq^{j+1}(5)$$

$$h = \frac{q^e(x) - q^j(x)}{q^{j+1}(x) - q^j(x)}.$$

The Brass method has been widely used and has formed the basis for many of the proposed methods for leveraging SBH data. The approach we describe next follows a different tack by beginning with probability models for fertility and mortality.

Preston, Heuveline, and Guillot (2000) provide a clear derivation of the Brass method that is based on a mean value theorem. The Brass approach provides neither a probability model that can be leveraged for smoothing nor an obvious way by which inference can be performed. Below we show that, under well-defined modeling assumptions and approximations, the Brass method can be seen as a method of moments.

2.2 An intuitive derivation of the new method

The approach is based on a plausible full probability model – i.e., a model that could be used to simulate birth and death data. We then consider taking a survey in which data

on births and deaths are gathered and describe a model for the data in the case when we have access to all birth and death times. We finally write down the model for the SBH observed data where we average over the unobserved data, and then approximate this form to obtain a computationally efficient model. We initially do this in the simplified case in which fertility and mortality do not change over time or with covariates.

Let $Y_m = 0/1$ be a binary indicator for a non-birth/birth when a woman is age m , with $Y_m | f_m \sim \text{Bernoulli}(f_m)$, where f_m is the probability of giving birth for a woman of age m for $m = m_{\min}, \dots, m_{\max}$. Following birth, we use a discrete time hazards model for death. For a child alive at age a , let $Z_a = 0/1$ be the binary indicator of non-death/death before age $a + 1$, given survival until age a . The distribution of the death indicators is $Z_a | {}_1q_a \sim \text{Bernoulli}({}_1q_a)$, where ${}_1q_a$ is the (conditional) probability of death (i.e., the hazard) between ages a and $a + 1$ for $a = 0, 1, \dots$. Define \mathbf{q} to be the vector containing these conditional probabilities of death. Now suppose that a survey is taken and M mothers of different ages are interviewed; without loss of generality, we focus on data from mothers of age m_s . Let $\mathbf{B}^{m_s} = [B_0, \dots, B_{m_s}]$ and $\mathbf{D}^{m_s} = [D_0, \dots, D_{m_s}]$ be the collections of births and deaths from these mothers. Here, B_a is the number of births born to women a years before the survey and D_a is the number of those children who died before the survey. These variables would be available for FBH data. The total numbers of births and deaths is defined as $T_B^{m_s} = \sum_{a=0}^{m_s} B_a$ and $T_D^{m_s} = \sum_{a=0}^{m_s} D_a$. These constitute the SBH data, but to derive a probability model for the data, we first write down a model for the full data.

For births a years before the survey, we assume the distribution of the number of deaths is

$$D_a | B_a, \mathbf{q} \sim \text{Binomial}(B_a, q(a)), \quad (1)$$

where $q(a) = 1 - \prod_{i=0}^{a-1} (1 - {}_1q_i)$ is the probability that a child born a years before the survey dies. A distribution for the collection of births is more difficult to write down (because the timings of births to the same mother are necessarily highly dependent), but as we will see, this will not be needed. Denote the birth distribution by $\Pr(\mathbf{B}^{m_s} | T_B^{m_s}, \mathbf{c})$ with $\mathbf{c} = [c_0, \dots, c_{m_s}]$ and where $c_a = f_a / \sum_{a'=0}^{m_s} f_{a'}$ is the probability that a woman gives birth a years before the survey, given that a birth occurs in the time interval (m_0, m_{m_s}) . Given the total number of births, the mean number of births a year before the survey is

$$E[B_a | T_B^{m_s}, c_a] = T_B^{m_s} c_a. \quad (2)$$

Hence we have forms upon which estimation of birth and death probabilities could be based, if we had complete data, as would be available in an FBH survey. To obtain a sampling model for the observed SBH data, we average over the missing information, first the unknown death times and then the unknown birth times.

When only the total number of deaths, $T_D^{m_s}$, is observed, the distribution of the sum of the deaths is a convolution of the distributions given in Equation (1). This is computationally intractable, since all possible legal combinations of deaths need to be enumerated; see Wakefield (2004). However, if we approximate the binomials by Poissons (which is valid if $q(a)$ is relatively small and the number B_a is large), we obtain a distribution for the total number of deaths as

$$T_D^{m_s} | \mathbf{B}^{m_s}, \mathbf{q} \sim \text{Poisson} \left(\sum_{a=0}^{m_s} B_a q(a) \right).$$

Let S_{m_s} be the set of legal configurations of births that can lead to a birth total of $T_B^{m_s}$. Averaging over this set of unknown births gives the mixture distribution:

$$T_D^{m_s} | T_B^{m_s}, \mathbf{c}, \mathbf{q} \sim \sum_{\mathbf{B}^{m_s} \in S_{m_s}} \Pr(\mathbf{B}^{m_s} | T_B^{m_s}, \mathbf{c}) \times \text{Poisson} \left(\sum_{a=0}^{m_s} B_a q(a) \right).$$

This expression looks complex, but it is quite intuitive, since the Poisson model describes the probability of the deaths, given a particular configuration of births, and these Poissons are averaged over the uncertainty in when the births occur. Unfortunately, this expression depends on the birth times to give the birth totals over time \mathbf{B}^{m_s} , and these are unobserved in the SBH data.

Rather than average over all elements in the sum, which is costly, we replace the distribution by the $m_s + 1$ means of $\Pr(\mathbf{B}^{m_s} | T_B^{m_s})$, which are given in Equation (2), to give

$$T_D^{m_s} | T_B^{m_s}, \mathbf{c}, \mathbf{q} \sim \text{Poisson} \left(T_B^{m_s} \sum_{a=0}^{m_s} c_a q(a) \right). \quad (3)$$

Hence we are conditioning on an estimate of the expectation of the number of births rather than averaging over the uncertainty in the distribution of births. This approximation will be most accurate when the number of births is large. To summarize, under a plausible sampling model, the distribution of the observed SBH data is a mixture of a convolution of binomials, but with a number of approximations we obtain a closed-form likelihood, namely Equation. (3), that is far more straightforward to work with. Within this likelihood, there are two sets of parameters to estimate: those associated with birth (\mathbf{c}) and those with death (\mathbf{q}).

From (3) we note that

$$\begin{aligned}
 E[T_D^{m_s} | T_B^{m_s}, \mathbf{c}, \mathbf{q}] &= E[E\{T_D^{m_s} | \mathbf{B}^{m_s}, \mathbf{c}, \mathbf{q}\}] \\
 &= E\left[\sum_{a=0}^{m_s} B_a q(a) | T_B^{m_s}, \mathbf{c}, \mathbf{q}\right] \\
 &= T_B^{m_s} \sum_{a=0}^{m_s} c_{m_s}(a) q(a),
 \end{aligned} \tag{4}$$

which is the expectation of Equation (3). The Brass method essentially treats Equation (4) as deterministic, replacing the left side with the observed number of deaths.

In this section we have shown that under a plausible and flexible model, the sampling model for the data available in SBH are a mixture of a convolution of binomial distributions, but by replacing the binomials with Poissons and approximating the distribution of births, we obtain a tractable sampling model. This form contains two sets of probabilities, $c(\cdot)$ for births and $q(\cdot)$ for deaths, and each of these may be modeled as functions of covariates and space and time, as we now describe.

2.3 The full model

The derivation in the last section will now be extended to the more realistic scenario in which the births and mortality models have greater complexity. We provide a summary here, with full details relegated to the supplementary materials. We specify models for fertility and mortality. Let $f_m(\mathbf{x}_t)$ denote the probability that a woman gives birth at age m and in year t , with \mathbf{x}_t containing the covariates at time t associated with birth. For mortality, we use a discrete hazards model. Let ${}_1q_a(\mathbf{x}_t) = q_a(1, \mathbf{x}_t)$ denote the risk of mortality – i.e., the probability that a child dies between age a and $a + 1$ – with \mathbf{x}_t now containing the covariates at time t associated with mortality. Although we use \mathbf{x}_t for covariates in both models for notational convenience, the covariates used for each will generally differ. The parameter of interest is $q(5, \mathbf{x}_t)$, the probability of death within five years of life, at time t and with covariates \mathbf{x}_t . This gives the probability of death before age 5,

$$q(5, \mathbf{x}_t) = 1 - \prod_{i=0}^4 (1 - {}_1q_i(\mathbf{x}_t)). \tag{5}$$

Let $q^*(a, \mathbf{x}_t)$ be the probability of dying within a years given birth at time t and with covariates \mathbf{x}_t ; thus

$$q^*(a, \mathbf{x}_t) = 1 - \prod_{i=0}^{a-1} (1 - q_i(\mathbf{x}_{t+i})). \quad (6)$$

The difference between Equations (5) and (6) for $a = 5$ is subtle, but crucial. In Equation (5), we envisage a synthetic cohort of children born in year t and then repeat year t five times at different ages, meaning they experience in the same calendar year the mortalities of each age band. In Equation (6), we instead imagine a real cohort of children who are born in year t and are followed up to year $t + 5$.

For women who are of age m_s years at the time of the survey, t_s , define $T_B^{m_s}(\mathbf{x}_t)$ to be the total number of children ever born to those women, with covariates \mathbf{x}_t at time t . Similarly, define $T_D^{m_s}(\mathbf{x}_t)$ to be the total number of children who ever died to women who are m_s at the time of the survey with covariates \mathbf{x}_t .

In the supplementary materials we derive a full likelihood, which is based on fertility and mortality models. Under a number of carefully defined simplifying approximations, we obtain a distribution for the observed deaths, $T_D^{m_s}$, given the observed births $T_B^{m_s}$ and dependent on birth parameters \mathbf{c} and mortality parameters \mathbf{q}^* :

$$T_D^{m_s} | T_B^{m_s}, \mathbf{c}, \mathbf{q}^* \sim \text{Poisson} \left(T_B^{m_s} \sum_{a=0}^{m_s} c_{m_s}(a, \mathbf{x}_{t_s-a}) q^*(a) \right), \quad (7)$$

where

$$c_{m_s}(a, \mathbf{x}_{t_s-a}) = \frac{f_{m_s-a}(\mathbf{x}_{t_s-a})}{\sum_{a=0}^{m_s} f_{m_s-a}(\mathbf{x}_{t_s-a})}.$$

Again, the total deaths is a mixture over the potential times at which the deaths could have occurred, with the mixture depending on the frequencies of births at different ages $c_{m_s}(a, \mathbf{x}_{t_s-a})$ (given a birth occurring) and the probabilities of dying at different ages $q^*(a)$. In practice, we specify models for the birth probabilities \mathbf{c} and the mortality parameters \mathbf{q}^* . Examples of such models are given in both the simulation study and the substantive application to estimating U5MR in Malawi.

Since \mathbf{c} is often unknown, we propose first fitting a fertility model to the FBH data. Such a model allows an estimate of $f(m, \mathbf{x}_t)$ to be formed – i.e., $\hat{f}(m, \mathbf{x}_t)$ – which can then be transformed to $\hat{c}_{m_s}(a, \mathbf{x}_t)$. In the simulation study that we next describe, we investigate the implications of the approximations embedded in (7).

3. Simulation study

In the simulation, we suppose there are two surveys that contain birth history information, both taken in 2010. One survey provides FBH information and a much larger survey (analogous to the census in our Malawi application) provides SBH information. We used the geography of Kenya, which is comprised of 47 regions (counties), labeled by $r = 1, \dots, 47$. In total, we simulated FBH data for $47 \times 4,000$ women and SBH data for $47 \times 20,000$ women, with equal numbers in each region. The supplementary materials give details on how the data were simulated and also the birth probabilities, f_m , used. These probabilities were set to be constant over five-year age groups of women and closely resemble patterns observed in the 2010 Malawi DHS (National Statistical Office – NSO/Malawi and ICF Macro 2011). The birth probabilities were also assumed constant over space and time. That is,

$$f_m(\mathbf{x}_t) = \frac{\exp(\beta_{c[m]})}{1 + \exp(\beta_{c[m]})},$$

with $c[m]$ an indicator function for mother's age m that takes on values: 1 if $m = 15, \dots, 19$, 2 if $m = 20, \dots, 24$, 3 if $m = 25, \dots, 29$, 4 if $m = 30, \dots, 34$, and 5 if $m = 35, \dots, 49$. Hence, we model the birth probabilities as being a five-level factor variable, with $\exp(\beta_c)$ being the odds of a birth in age band c , for $c = 1, \dots, 5$. We simulate deaths using three distinct discrete hazards: one for the first year of life, one for ages 1, 2, 3, 4, and one for age 5 onward. Each of these are a function of time (five-year periods) t and region r , so that

$${}_1q_a(t, r) = \frac{\exp(\beta_{c[a]} + \phi_{c[a]}(t) + S_r + \epsilon_r)}{1 + \exp(\beta_{c[a]} + \phi_{c[a]}(t) + S_r + \epsilon_r)},$$

where $c[a]$ takes the values 1 for $\{a = 0\}$, 2 for $\{a = 1, \dots, 4\}$, and 3 for $\{a = 5, \dots\}$. The interpretation of the parameters is as follows:

- $\exp(\beta_c)$ are the odds of death at time 0 and are taken as fixed effects, with $c = 1, 2, 3$, to allow the age curves to start from three different intercepts.
- $\exp(\phi_c(t))$ describe how the odds of death in age band c change across years, via a smoothing model. Specifically, the three $\phi_c(t)$ terms ($c = 1, 2, 3$) are modeled as second-order random walks (RW2) over time (Rue and Held 2005). In terms of

second differences:

$$\underbrace{[\phi_c(t) - \phi_c(t-1)]}_{\text{Slope between } t \text{ and } t-1} - \underbrace{[\phi_c(t-1) - \phi_c(t-2)]}_{\text{Slope between } t-1 \text{ and } t-2} \sim N(0, 1/\kappa_T), \quad (8)$$

showing that deviations from linearity are being modeled. That is, this model encourages a linear trend for the log odds but allows fluctuations if the data suggest these are warranted. The model has a single smoothing parameter, the precision κ_T , with large values giving smoother trajectories. In Equation (8), if κ_T is large, the variance is small and so the local slopes are more tightly tied together, meaning they are similar in magnitude. We emphasize that the parameter κ_T is assigned a prior and then estimated from the data, so that an appropriate amount of smoothing is applied.

- $\exp(S_r)$ describe how the odds of death vary smoothly across geographical regions r . The spatial smoothness is modeled via an intrinsic conditional autoregressive (ICAR) model (Rue and Held 2005). Under this model, the spatial effects S_r are modeled conditional on the neighbors. Specifically,

$$S_r | \{S_{r'} = s_{r'}, r' \sim r\}, \kappa_S \sim N\left(\bar{s}_r, \frac{1}{\kappa_S m_r}\right),$$

where $r' \sim r$ is shorthand for r' is a neighbor of r , and where we define a neighboring area as one that shares a boundary, $\bar{s}_r = \frac{1}{m_r} \sum_{r' \sim r} s_{r'}$ is the mean of the neighbors of area r , and m_r is the number of such neighbors. The effect of this prior is to pull mortality risks toward neighboring risks, so that κ_S is a spatial smoothing parameter with large values indicating little spatial smoothing.

- $\exp(\epsilon_r)$ is an independent odds contribution at the region level that corresponds to a “random shock” in that there is no spatial structure. The random regional shocks are modeled as $\epsilon_r \sim_{iid} N(0, 1/\kappa_\epsilon)$ with iid shorthand for “independent and identically distributed.” Hence, if κ_ϵ is large, there are small random shocks only.

Together, S_r and ϵ_r correspond to the celebrated Besag, York, Mollié (BYM) model, which was introduced in Besag, York, and Mollié (1991) and is the most commonly used model in spatial epidemiological studies. To summarize, both the RW2 and ICAR terms encourage local smoothness in the mortality risk in time and space, respectively. Each has a sum-to-zero constraint for identifiability. For the simulation we used values of $\phi_c(t)$ that were similar to those observed in our Malawi application, and we simulated the structured (S_r) and unstructured (ϵ_r) random effects to produce spatial risk surfaces.

We examine model-fitting when three different types of birth information are available (see supplementary materials for more details):

1. The true number of SBH births by age is known, $B_{m_s}(a)$, and included in the Poisson model.
2. The true birth probabilities, $c_{m_s}(a)$, are known and the approximate Poisson model (Equation (7)) is used to include the SBH data.
3. The birth probabilities are estimated from the FBH data via logistic regression, and the estimates $\hat{c}_{m_s}(a)$ are plugged in for $c_{m_s}(a)$ in the approximate Poisson model (Equation (7)).

The supplementary materials include a plot of the observed fertility in the SBH data, along with the various approximations. At all ages we see that the results are almost identical under the different model-fitting strategies.

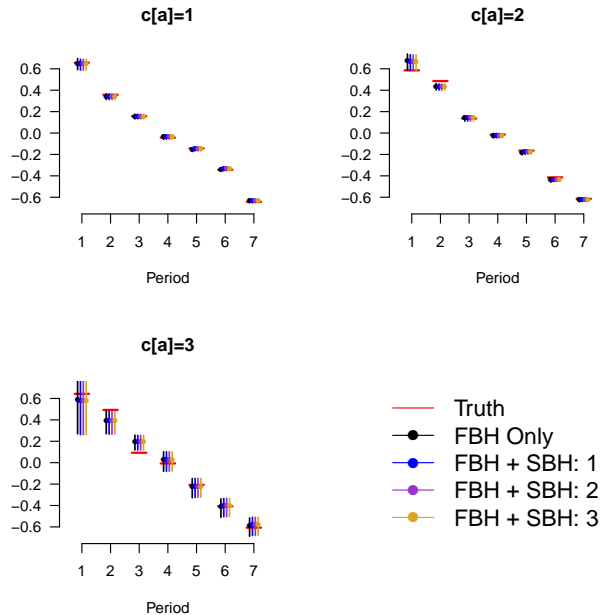
Table 1 reports the true parameter values and the estimates and uncertainty for the intercepts and smoothing parameters under an FBH-only analysis and under combined FBH + SBH analyses in which different information was used on births/fertilities. Figure 1 displays how the random walk temporal smoothing estimates for mortality, $\hat{\phi}_c(t)$, evolve over time t under the different approaches. The three approaches to incorporating SBH yield very similar results and are comparable to the FBH-only summaries. There are some identifiability problems with the two spatial random effects S_r and ϵ_r (which is a common phenomena in spatial analyses and not a problem when prediction of the overall mean, rather than the separate components, is all that is required); hence we display results for $S_r + \epsilon_r$ in the supplementary materials.

Table 1: Simulation study: Comparison of estimates and 95% uncertainty intervals when using FBH-only and FBH + SBH data

	Truth	FBH Only	FBH + SBH: 1	FBH + SBH: 2	FBH + SBH: 3
$\exp(\beta_1)$	0.150	0.148 (0.144, 0.153)	0.148 (0.143, 0.153)	0.148 (0.143, 0.153)	0.148 (0.143, 0.153)
$\exp(\beta_2)$	0.053	0.054 (0.052, 0.055)	0.053 (0.051, 0.055)	0.053 (0.051, 0.055)	0.053 (0.051, 0.055)
$\exp(\beta_3)$	0.005	0.005 (0.005, 0.006)	0.005 (0.005, 0.006)	0.005 (0.005, 0.006)	0.005 (0.005, 0.006)
κ_T	—	200 (73, 542)	199 (75, 531)	197 (73, 539)	198 (74, 539)
κ_S	45	34 (3.5, 313)	38 (4.0, 342)	37 (3.9, 350)	37 (3.7, 365)
κ_ϵ	90	103 (3.85, 2760)	86 (5.11, 1540)	86 (4.8, 1540)	87 (5.1, 1510)

Note: SBH data is incorporated using one of the three approaches: (1) true births, (2) true fertilities, (3) estimated fertilities. The exponentiated intercepts (odds ratios) are $\exp(\beta_1)$, $\exp(\beta_2)$, and $\exp(\beta_3)$ and the precision parameters for time, space, and iid terms are κ_T , κ_S , and κ_ϵ . The time trend was taken to mimic the Malawi data (so that we do not simulate from an RW2 model) and so there is no true precision parameter on the random walk.

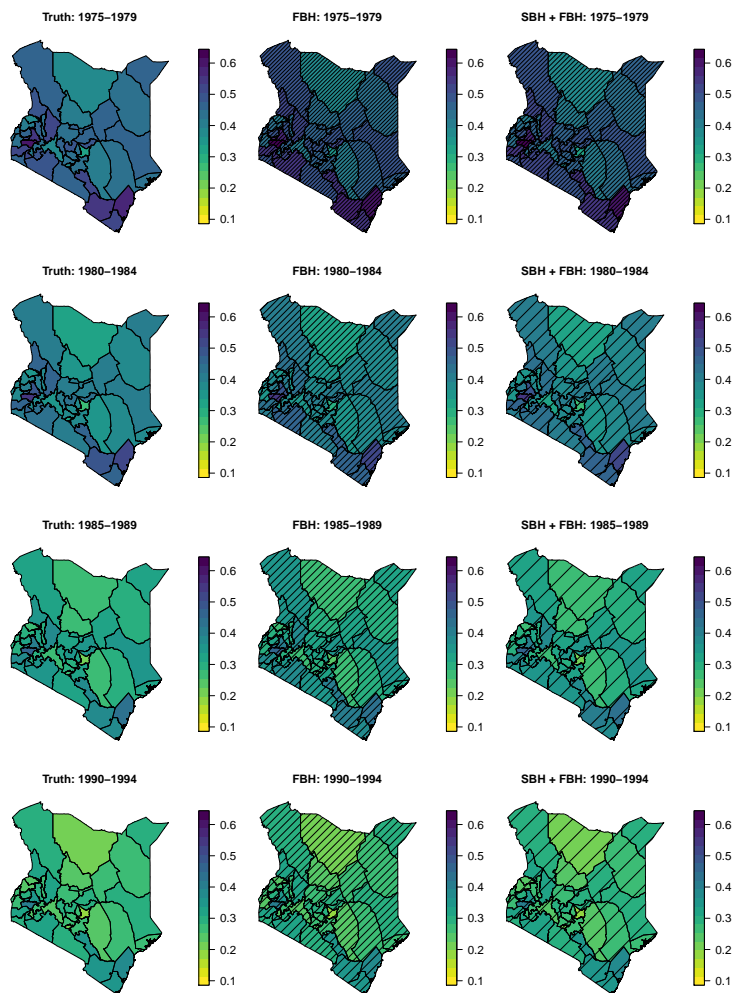
Figure 1: Simulation study: Comparison of RW2 parameters when using FBH-only and FBH + SBH Data



Note: Posterior median estimates of RW2 parameters, $\phi_{c[a]}(t)$, in mortality model, with 95% uncertainty intervals, for three different age groups $c[a]$ (0–1, 1–5, 5+). We include the truth, FBH-only, and three approaches for dealing with the SBH data: (1) true births, (2) true birth probabilities, (3) estimated birth probabilities.

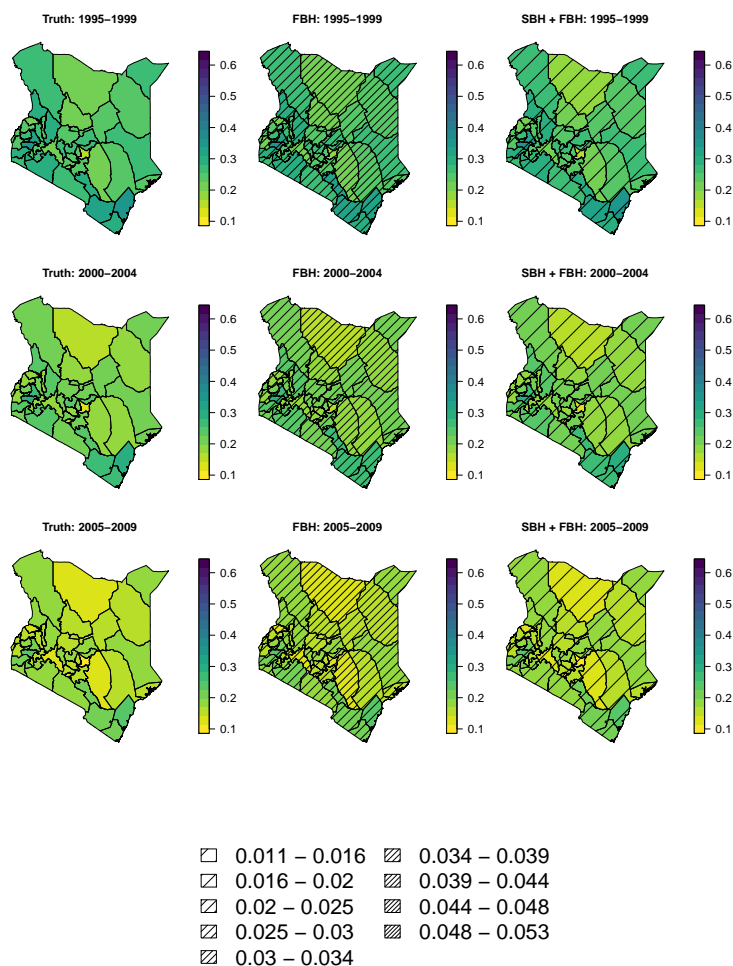
The mortality hazard can be combined to give U5MR estimates over time via Equation (5). Figures 2 and 3 visually depict estimates of U5MR with uncertainty (conveyed via the standard deviation), expressed using hatching, with denser hatching reflecting greater uncertainty. See the supplementary materials for more details on how these estimates were obtained. For presentation purposes, “SBH + FBH” refers to using the third approach for including the SBH data (i.e., plugging in the fertility estimates from the FBH-only analysis). Both the FBH and SBH + FBH give similar estimates for U5MR and are close to the truth, but we clearly see that the uncertainty is reduced when SBH data are incorporated, as expected. For reference, the estimated standard deviation (on the logit U5MR scale) in 1975–1979 was 10%–21% (mean 14%) higher than when only FBH data were used. In 2005–2009, the estimated standard deviation was 90%–120% (mean 105%) higher when only FBH data were used, as compared to the analysis with FBH + SBH data.

Figure 2: Simulation study: Truth and estimates of U5MR in the first four periods



Note: True and estimates of U5MR in the first four periods by Kenyan counties. Uncertainty (standard deviation of logit U5MR) is expressed through hatching with denser hatching indicating greater uncertainty. Note the wider hatching in the right-hand column as compared to the middle column, which shows the added benefit of the SBH data. The hatching legend is given in Figure 3.

Figure 3: Simulation study: Truth and estimates of U5MR in the last three periods



Note: True and estimates of U5MR in the most recent three periods by Kenyan counties. Uncertainty (standard deviation of logit U5MR) is expressed through hatching with denser hatching indicating greater uncertainty. Note the wider hatching in the right-hand column as compared to the middle column, which shows the added benefit of the SBH data.

We conclude from this simulation that the approximations perform well in general and that the SBH data can produce more precise estimates when used to supplement FBH data than when we use estimates from FBH data alone.

4. Application to Malawi data

4.1 Context

In the five-year period before the 1992 Malawi DHS, the national U5MR was 234 deaths per 1,000 live births, and this fell to 63 deaths per 1,000 in the five-year period prior to the 2015-16 DHS (Malawi DHS 2016). This impressive decrease hides significant subnational variation, however. Using simple weighted estimation techniques, the most recent DHS and Multiple Indicator Cluster Surveys (MICS) reported summaries for three subregions of Malawi. Table CM.2 of Malawi MICS (2015) gives U5MR estimates for northern, central, and southern regions (over the five years prior to the survey) as 67, 81, and 92, respectively. Similarly, in Table 8.2 of Malawi DHS (2016) the U5MR estimates for northern, central, and southern regions (over the 10 years prior to the survey) are 57, 81, and 73. Even between these two surveys we see inconsistencies of estimates in the southern region (though this is a little hard to judge without examination of uncertainty intervals).

In this analysis we estimate five-year U5MR in each of the districts of Malawi, using FBH and SBH data. We note that although there are 28 districts, due to the sampling done in the 2006 MICS we combine some of the districts so that we end up modeling 26 regions. Specifically, Mwanza and Neno were combined into a single region, as were Nkhata Bay and Likoma.

4.2 Data

The available data include three DHS, taken in 2004, 2010, and 2015; two MICS, taken in 2006 and 2013; and a census taken in 2008. All five surveys contain FBH data, whereas the census contains SBH data only. Microdata, from which we can access the required geographic information, is available on a 10% random sample of the census. Table 2 summarizes the numbers of clusters, women, births, and deaths for the Malawi data. We see the census data dominates in terms of women, births and deaths, but the lack of times on births and deaths means the information content of these data is far less than that provided by the FBH data. The cluster surveys are stratified by geographical area and urban/rural. The latter means it is important to include a term for urban/rural in the model to account for the differential sampling of urban and rural clusters (for example, in the 2015 Malawi DHS, urban clusters were almost twice as likely to be sampled as rural

clusters), and the differences in mortality between urban and rural areas. However, urban residence is defined at the time of survey, not at the time of exposure to risk of births or deaths, and so for periods well before a survey, any urban effect is likely to be diluted by migration and urbanization.

Table 2: Application: Data summaries

Survey	# Clusters	# Women	# Births	# Deaths
DHS 2004	521	11,698	35,883	6,534
MICS 2006	1,040	26,211	78,641	11,855
Census 2008	—	307,471	871,327	150,432
DHS 2010	849	23,020	72,301	11,343
MICS 2013	1,139	24,220	72,568	9,213
DHS 2015	850	24,562	68,074	7,235

Note: SBH data is available from the census, and FBH data is from all other data sources (which are all household surveys).

In the supplementary materials we present results from an exploratory analysis, comparing the five surveys and census. These results suggest that women in the census tend to report numbers of children born that are similar to numbers in the surveys. However, women in rural areas tended to report that more of their children died. This observation motivated inclusion of a bias term by urban/rural in our mortality model.

The fertility model we use is

$$f(m, \mathbf{x}_t) = \frac{\exp(\beta_{\text{RUR}} \times 1(\text{rural}) + \beta_{\text{URB}} \times 1(\text{urban}) + \phi(p) + \phi(m) + S_r + \epsilon_r)}{1 + \exp(\beta_{\text{RUR}} \times 1(\text{rural}) + \beta_{\text{URB}} \times 1(\text{urban}) + \phi(p) + \phi(m) + S_r + \epsilon_r)},$$

where β_{RUR} and β_{URB} are fixed effects for strata (rural/urban), with $1(\text{rural})$ and $1(\text{urban})$ being an indicator for whether the mother resides in an area classified as rural or urban, respectively, $\phi(p)$ is a RW2 over roughly five-year time periods p , $\phi(m)$ is a RW2 over age, S_r is an ICAR spatial random effect, and ϵ_r is an unstructured (iid) error on regions r that allows for independent “shocks.”

The mortality model is

$$1q_a(\mathbf{x}_t) = \frac{\text{HIV}(p) \exp(\eta)}{1 + \text{HIV}(p) \exp(\eta)},$$

where

$$\eta = \beta_{\text{URB}} \times 1(\text{urban}) + \beta_{\text{SBH}} \times 1(\text{SBH}) + \beta_{\text{SBH,URB}} \times 1(\text{SBH and urban}) + \beta_{b[a]} + \text{smooth}$$

and

$$\text{smooth} = \phi_{c[a]}(p) + S_r + \epsilon_r.$$

This model therefore has mortality being different in urban and rural areas and allows for systematic bias in the SBH data as compared to the FBH data (which we are assuming are less likely to be biased), and the size of the bias can be different in urban and rural areas. These associations are assumed to be constant across region and time period so that the overall levels are not informed by the SBH data but rather the data inform on the spatial and temporal differences.

Age-specific intercepts and age-specific random walks are indexed by

$$b[a] = \begin{cases} 1 & a = 0, \\ 2 & a = 1, \\ 3 & a = 2, \\ 4 & a = 3, \\ 5 & a = 4 \\ 6 & a = 5, 6, \dots \end{cases} \quad c[a] = \begin{cases} 1 & a = 0, \\ 2 & a = 1, \dots, 4 \\ 3 & a = 5, 6, \dots \end{cases}$$

respectively. So the curves start from six different points (defined by different ages) but subsequently follow three curves over time; ages 1, 2, 3, and 4 share a trajectory, for reasons of parsimony and based on initial analyses. Hence these four age trajectories are parallel. We again include ICAR spatial random effects S_r for region along with iid errors ϵ_r . Although we use the same letters for these spatial terms in both fertility and mortality models, we stress that we have separate spatial and iid terms for each model. The fixed effects $\beta_{b[a]}$ are yearly intercepts for the first five years, with an additional term for all ages greater than 5. If there were interest in yearly mortality beyond 5, we would include yearly intercepts for this range also (though deaths become much rarer after age 5). The data for Malawi are relatively extensive, so we do not smooth over age. In other situations in which there were less data it would be natural, and straightforward, to place a smoothing prior – for example an RW2 – on the $\beta_{b[a]}$ parameters or to place informative priors on these coefficients. We include an HIV adjustment, as in previous work with FBH data (Wakefield et al. 2019). Briefly, we wish to adjust for the loss of child deaths due to mothers who have died of AIDS; the children of these mothers are more likely to die, and so the missingness is informative. We estimate the bias using the method described in Walker, Hill, and Zhao (2012), which uses a cohort component projection model.

4.3 Results

In the supplementary materials we give parameter summaries and visual summaries for fertility. For mortality, Table 3 gives the parameter estimates. As expected, mortality decreases monotonically with age, as indicated by the odds ratio estimates, $\exp(\beta_0)$ – $\exp(\beta_5)$. Mortality is decreased in urban areas, with the estimate from the FBH data alone indicating a 29% decrease. The SBH data give 19% higher mortality estimates in rural areas and approximately the same mortality estimates in urban areas as found in the FBH data. This finding was confirmed by examination of the raw data (see supplementary materials).

Table 3: Application: Comparison of parameter estimates and 95% uncertainty intervals when using FBH only and SBH + FBH data

	FBH Only	FBH + SBH
$\exp(\beta_1)$	0.103 (0.097, 0.109)	0.100 (0.093, 0.107)
$\exp(\beta_2)$	0.033 (0.031, 0.036)	0.030 (0.027, 0.033)
$\exp(\beta_3)$	0.025 (0.023, 0.027)	0.022 (0.020, 0.024)
$\exp(\beta_4)$	0.018 (0.016, 0.019)	0.015 (0.014, 0.017)
$\exp(\beta_5)$	0.009 (0.008, 0.010)	0.008 (0.007, 0.009)
$\exp(\beta_6)$	0.005 (0.004, 0.006)	0.005 (0.004, 0.006)
$\exp(\beta_{\text{URB}})$	0.710 (0.686, 0.736)	0.716 (0.692, 0.741)
$\exp(\beta_{\text{SBH}})$	–	1.185 (1.171, 1.200)
$\exp(\beta_{\text{SBH,URB}})$	–	0.857 (0.823, 0.891)
$\kappa_{\mathcal{T}}$	24.9 (11.0, 57.1)	15.5 (6.74, 35.3)
$\kappa_{\mathcal{S}}$	123 (52.1, 291)	153 (59.2, 392)
κ_{ϵ}	322 (118, 888)	252 (80.5, 784)

Note: Exponentiated intercepts for the age trends are $\exp(\beta_1), \dots, \exp(\beta_6)$. Exponentiated intercept for urbanicity is $\exp(\beta_{\text{URB}})$. Exponentiated bias terms for the SBH data by urban/rural are $\exp(\beta_{\text{SBH}})$ and $\exp(\beta_{\text{SBH,URB}})$. Precision for the time, space, and iid terms are $\kappa_{\mathcal{T}}$, $\kappa_{\mathcal{S}}$, and κ_{ϵ} .

The extent of the spatial and temporal smoothing can be determined by examination of the relevant smoothing parameters (though care is required since these are conditional parameters and not directly comparable), with small values of the precisions (inverse variances) corresponding to stronger trends. Here we see that the time trends are very strong (low precision for RW2 model, $\kappa_{\mathcal{T}}$).

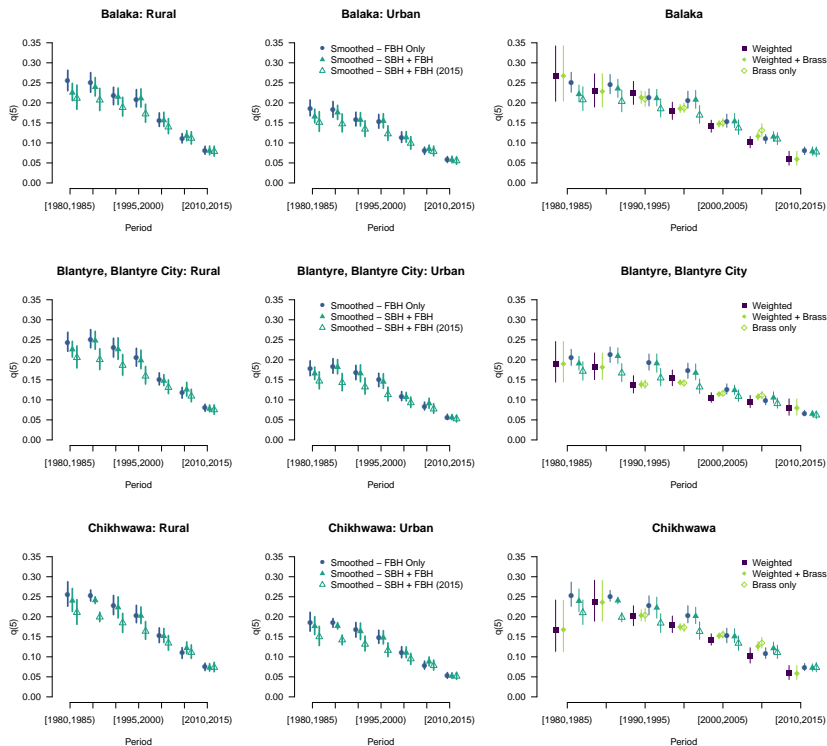
Figure 4 compares U5MR results for three regions, using our model when only fit to FBH data and when we use our new approach to add in the SBH data. Results for the other 23 regions can be found in the supplementary materials. In this example, we have extensive survey data from DHS and MICS. In this sense, this example is not typical. Therefore, to examine how our approach might perform in settings with less FBH data,

we attempted to run the model using the most recent DHS only (to provide the FBH data) along with the census SBH data. We found that a single DHS did not provide enough data to sufficiently tie down the fertility model (which is only fit on FBH data), and thus we used the fertility model from all surveys. However, we were able to fit the mortality model using only the most recent DHS in conjunction with the SBH data, and the results are shown in Figure 4, labeled as “SBH + FBH (2015).” Focusing on the urban/rural estimates, the SBH + FBH (2015) U5MR estimates tend to be even lower in the earlier time periods and are slightly lower as compared to the SBH + FBH (all surveys) analysis in the most recent time periods. We hypothesize that this is happening because we are using only the 2015 DHS, and these women tended to report a smaller CD/CEB ratio (see Figure 10), thus U5MR would be lower.

The flip side of the Brass method providing only a very basic U5MR summary is that it does not rely on much input information. By contrast, our model-based approach provides rich spatiotemporal summaries for mortality and fertility but requires more detailed data. So, for example, while we could attempt to produce inference for fertility and mortality from SBH data alone, we would not recommend attempting this. In many situations additional information will be available to estimate fertility. Here we use information from all the DHS surveys to estimate the fertility part of the model.

Finally, we aggregate the results stratified by urban/rural up to the region level and compare to the direct estimates, estimates based only on using the Brass method applied to the SBH data, and the direct estimates combined with Brass. In general, results are fairly similar for all methods. Overall, U5MR decreases over time. Comparing FBH only and SBH + FBH results, U5MR tends to be lower in earlier time periods and higher in more recent time periods when SBH is included. Comparing to the case when we include only the most recent DHS – i.e., SBH + FBH (2015) – we observe the same pattern as we did with the urban/rural estimates. U5MR tends to be lower than when all FBH surveys are included, with a bigger difference in the earlier time periods. We also observe that the Brass-only estimates tend to be higher in the 2005–2009 period, which is expected since these estimates are based on the youngest women. Thus adding in SBH data with the Brass method also tends to result in higher U5MR in more recent time periods (as compared to results from only the direct estimates). The effect of the HIV epidemic can be seen in the slowing down of the decline in U5MR in the 1990s. The temporal trends are consistent with national trends shown in Figure CM.3 of Malawi MICS (2015). In many time periods the Brass estimates have fairly small uncertainty; the uncertainty looks too low here (for example, comparing Brass estimates in consecutive periods). A thorough simulation would need to confirm this by examining the frequentist properties of the interval estimates, but there is no reason to expect the correct coverage since the Brass method is not constructed using a formal statistical model and estimation procedure.

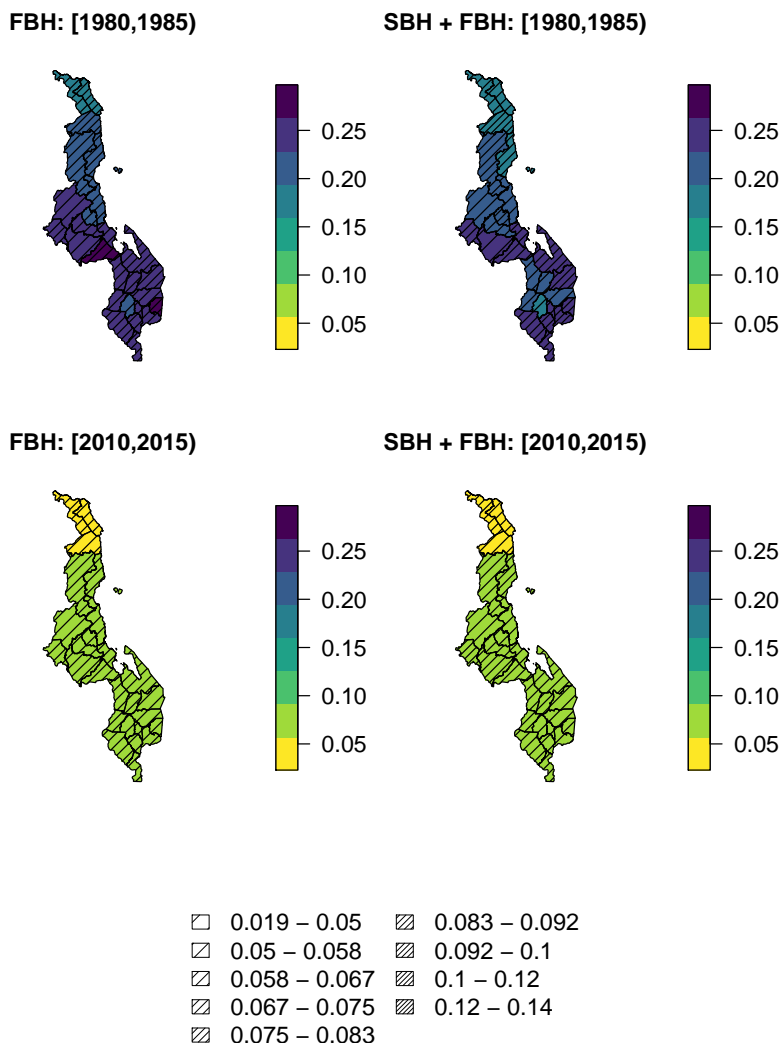
Figure 4: Application: Posterior medians and 95% uncertainty intervals for U5MR, stratified by urban and rural for 3 regions



Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

We now return to comparison of estimates obtained from all FBH data and all FBH data plus SBH data. Maps of the posterior median of U5MR are shown over time in Figure 5 (maps for the other time periods are in the supplementary materials). The density of the hatching corresponds to the standard deviation in (logit) U5MR, with denser hatching reflecting greater uncertainty. In general, we observe similar trends using FBH data alone and FBH + SBH, and we observe a decrease in uncertainty over time. Overall, uncertainty is reduced when the SBH data are included.

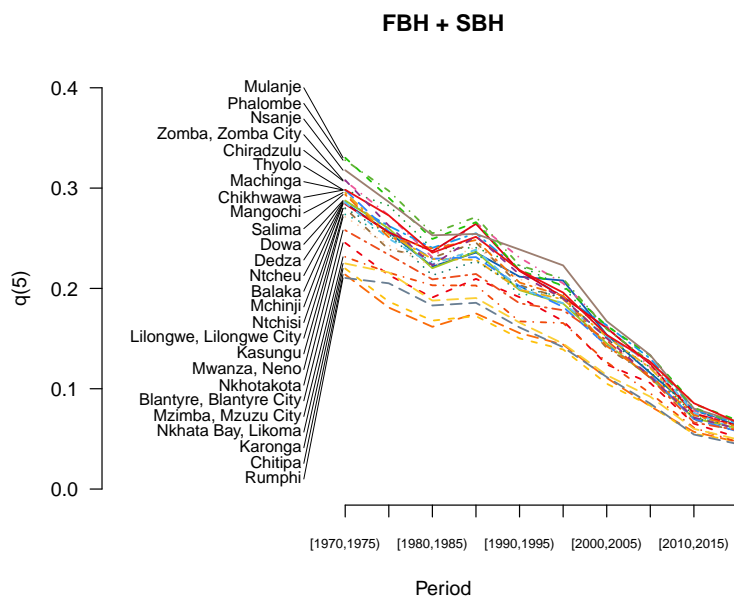
Figure 5: Application: Maps of U5MR (posterior medians) for the time periods 1980–1985 and 2010–2015



Note: Left: using FBH data only. Right: incorporating SBH data. Denser hatching reflects greater uncertainty (based on the posterior standard deviation of logit U5MR).

Figure 6 provides the trends in U5MR (posterior median) over time for the 26 regions from the FBH + SBH model. The figure shows a decrease in all regions until the 1980s, some leveling, and then a decrease. There is little crossing over of the curves, which shows that the interaction between space and time is not a significant contribution. In the SBH + FBH model for the time period 2010–2015, the regions with the lowest U5MR (posterior median less than 57 per 1,000) were Rumphi, Karonga, and Chitipa, the three northernmost regions. The regions with highest U5MR (posterior median greater than 80 per 1,000) were Nsanje, Phalombe, Thyolo, and Mulanje, the four southeastern regions. Overall, we see that the absolute subnational variation in U5MR has decreased over time, but in relative terms there is still significant subnational variation. Specifically, the overall range for 2010–2015 was 53 (Rumphi) to 88 (Mulanje), and for the period 1970–1975 the range was 210 (Rumphi) to 330 (Mulanje). So although there has been a dramatic decline, the relative range of (largest-smallest)/smallest is about 0.6 in both periods. That is, the area with the highest U5MR is 60% greater than the lowest. This level of spatial detail is much greater than is available in the DHS and MICS reports.

Figure 6: Application: Trends of U5MR over time for the 26 regions in Malawi



Note: Posterior medians from the combined FBH + SBH model.

5. Discussion

In this paper we have described a flexible model for U5MR estimation that combines FBH and SBH data in an efficient implementation. Code to implement the simulation study is available at <https://github.com/wilsonka/SBH>. The Malawi analysis took seven minutes to run on a laptop.

The method in the paper has been developed to *combine* FBH and SBH data, rather than as an alternative to Brass, in the context of subnational estimation. In general, trying to apply Brass to obtain yearly subnational estimates is likely to fail because of the paucity of information at this space and time granularity. By assuming a model-based approach we are able to apportion the information at more granular spatial and temporal levels and to estimate the associated uncertainty in a more statistically rigorous way. The method also allows the systematic assessment of differences between the two sources of information and allows us to see if these differences are associated with covariates such as measures of urbanicity.

Since we have specific models for both fertility and mortality, it is straightforward to include spatial and temporal smoothing models for each, and of a type that the context requires (or the user is familiar with). We used random walk and intrinsic CAR models, but it is straightforward to use other forms, such as temporal spline models (Alkema and New 2014) or the Leroux spatial model (Riebler et al. 2016). We focused on U5MR, but mortality estimates for ages under 5 are natural (and easy) to obtain, given our discrete hazards mortality model. We could also include covariates in our models. For example, mother's education has been found to be associated with U5MR (Golding et al. 2017).

We described the model using mother's age, but the approach could be extended if time since first birth or time since marriage (Hill, Zlotnik, and Trussell 1983; Hill and Figueroa 1999) were deemed preferable time scales. Currently, national estimates of U5MR produced by the United Nations incorporate SBH data using indirect estimates obtained from the time since first birth variant (Alkema et al. 2014; Hill et al. 2012).

A major difficulty that has long plagued the synthesis of FBH and SBH data is reconciling systematic differences between the two data sources. Wilson and Wakefield (2021) found systematic differences in data from Malawi, and differences have also been found in other applications (Hill et al. 2015). In our example and in Wilson and Wakefield (2021) we have attributed the differences to biases in the SBH data, but biases also exist in FBH data from household surveys and some surveys are not used because they are thought to be unreliable. For examples, see the supplementary materials of Li et al. (2019), in which there is a list of surveys that were not used in estimation of the U5MR in sub-Saharan Africa. However, where local and contextual information is available, the model we have proposed allows the flexible modeling of biases in either data source in a transparent manner. Theoretically, we could analyze SBH data alone, if we had relevant fertilities to use in the model, but this endeavor is inherently dangerous due to the close to

non-identifiability of the SBH data alone. We are actively working on methods for using SBH data that reduce the reliance on FBH data.

In conclusion, while there are many challenges to modeling SBH data, the Brass method has proved very useful over the many years since its introduction, and the modeling approach we describe provides the potential to leverage SBH data to an even greater extent.

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Appendix

Further details on the Brass method

Table A-1: Correspondence between mothers age group and mortality age, as used in the Brass method

Age group	i	x
15–19	1	1
20–24	2	2
25–29	3	3
30–34	4	5
35–39	5	10
40–44	6	15
45–49	7	20

Note: The probability of death before age x is defined as $q(x)$. The mother's age group is indexed by $i = 1, \dots, 7$, and x is the corresponding age of children for whom cumulative mortality is best identified by the proportion of children who died to mothers in age group i . So, for example, the proportion of children who died to mothers in the age range 30–34 ($i = 4$) estimates $q(5)$, the U5MR.

Further details on the full model

Here we extend the simplified derivation described in “An intuitive derivation of the new method” to the more realistic scenario in which the births and mortality models are more complex. Let $f_m(\mathbf{x}_t)$ denote the probability that a woman gives birth at age m and in year t , with \mathbf{x}_t containing the covariates at time t associated with birth. For mortality, we use a discrete hazards model. Let ${}_1q_a(\mathbf{x}_t) = q_a(1, \mathbf{x}_t)$ denote the risk of mortality – i.e., the probability that a child dies between age a and $a + 1$, with \mathbf{x}_t now containing the covariates at time t associated with mortality. Although we use \mathbf{x}_t for covariates in both models for notational convenience, the covariates used for each will generally differ. The parameter of interest is $q(5, \mathbf{x}_t)$, the probability of death within five years of life at time t and with covariates \mathbf{x}_t . Here,

$$q(5, \mathbf{x}_t) = 1 - \prod_{i=0}^4 (1 - {}_1q_i(\mathbf{x}_t)). \quad (9)$$

Let $q^*(a, \mathbf{x}_t)$ be the probability of dying within a years *given birth at time t* and with covariates \mathbf{x}_t ; thus,

$$q^*(a, \mathbf{x}_t) = 1 - \prod_{i=0}^{a-1} (1 - {}_1q_i(\mathbf{x}_{t+i})). \quad (10)$$

The difference between Equations (9) and (10) for $a = 5$ is subtle. In Equation (5), we envisage a synthetic cohort of children who are born in year t and then repeat year t five times, meaning they experience in the same calendar year the mortalities of each age band. In Equation (10), we instead imagine a real cohort of children who are born in year t and are followed up to year $t + 5$.

For FBH data, where information is available on when births and deaths occurred, let Y_{mt} be an indicator for birth in year t to a woman of age m years (in the case of multiple births, we include multiple indicators). Let Z_{at} be an indicator for death between ages a and $a + 1$ in years t to $t + 1$. To summarize, a reasonable model for the FBH data consists of the birth and death components:

$$\begin{aligned} Y_{mt} | f(m, \mathbf{x}_t) &\sim \text{Bernoulli}(f_m(\mathbf{x}_t)), \\ Z_{at} | {}_1q_a(\mathbf{x}_t) &\sim \text{Bernoulli}({}_1q_a(\mathbf{x}_t)). \end{aligned}$$

Now consider SBH data from a survey taken in year t_s . For women of age m_s years at the time of the survey, define $T_B^{m_s}(\mathbf{x}_t)$ to be the total number of children ever born to those women who have covariates \mathbf{x}_t for all t . Further, for these women define the (unobserved) number of children born a years prior to the survey as $B_a(\mathbf{x}_{t_s-a})$. Note that $\sum_{a=0}^{m_s} B_a(\mathbf{x}_{t_s-a}) = T_B^{m_s}(\mathbf{x}_t)$. Similarly, define $T_D^{m_s}(\mathbf{x}_t)$ to be the total number of children who ever died to women who are m_s at the time of the survey who have covariates \mathbf{x}_t for all t . Define the (unobserved) number of children who were born a years prior to the survey *and* died by the time of the survey (i.e., died within a years) to be $D_a(\mathbf{x}_{t_s-a})$. Again, $\sum_{a=0}^{m_s} D_a(\mathbf{x}_{t_s-a}) = T_D^{m_s}(\mathbf{x}_t)$. Therefore, a reasonable model for the unobserved data is,

$$D_a(\mathbf{x}_t) | B_a(\mathbf{x}_t), q^*(a, \mathbf{x}_t) \sim \text{Binomial}(B_a(\mathbf{x}_t), q^*(a, \mathbf{x}_t)),$$

which provides a starting point for deriving an approximation to the distribution of the SBH data. Approximating the binomial with a Poisson,

$$D_a(\mathbf{x}_t) | B_a(\mathbf{x}_t), q^*(a, \mathbf{x}_t) \sim \text{Poisson}(B_a(\mathbf{x}_t)q^*(a, \mathbf{x}_t)).$$

Suppressing the dependence on \mathbf{x}_t , for notational convenience, and summing over age gives,

$$T_D^{m_s} | \mathbf{B}^{m_s}, q^*(a) \sim \text{Poisson} \left(\sum_{a=0}^{m_s} B_a q^*(a) \right). \quad (11)$$

Finally, we sum over all $(m_s + 1)^{T_B^{m_s}}$ possible combinations of when births could have occurred for a given $T_B^{m_s}$ to obtain a mixture distribution. We let S_{m_s} represent the set of all legal configurations of births that can sum to $T_B^{m_s}$. Then,

$$T_D^{m_s} | T_B^{m_s}, \mathbf{f}_{m_s}, \mathbf{q}^*(a) \sim \sum_{\mathbf{B} \in S_{m_s}} \Pr(\mathbf{B}^{m_s} | T_B^{m_s}, \mathbf{f}_{m_s}) \times \text{Poisson} \left(\sum_{a=0}^{m_s} B_a q^*(a) \right), \quad (12)$$

where \mathbf{f}_{m_s} are the vector of birth probabilities prior to the survey for a woman of age m_s at the survey.

We approximate Equation (12) by

$$T_D^{m_s} | T_B^{m_s}, \mathbf{c}_{m_s}, \mathbf{q}^*(a) \sim \text{Poisson} \left(T_B^{m_s} \sum_{a=0}^{m_s} c_{m_s}(a) q^*(a) \right), \quad (13)$$

where

$$c_{m_s}(a, \mathbf{x}_{t_s-a}) = \frac{f_{m_s-a}(\mathbf{x}_{t_s-a})}{\sum_{a=0}^{m_s} f_{m_s-a}(\mathbf{x}_{t_s-a})}.$$

Since \mathbf{c} is often unknown, we propose first fitting a fertility model to the FBH data. Such a model allows an estimate of $f(m, \mathbf{x}_t)$ to be formed. This estimate, $\hat{f}(m, \mathbf{x}_t)$, can then be transformed to $\hat{c}_{m_s}(a, \mathbf{x}_t)$. In the simulation study that we next describe, we investigate the effect of both replacing Equation (12) with Equation (13) and compare with estimating \mathbf{c} .

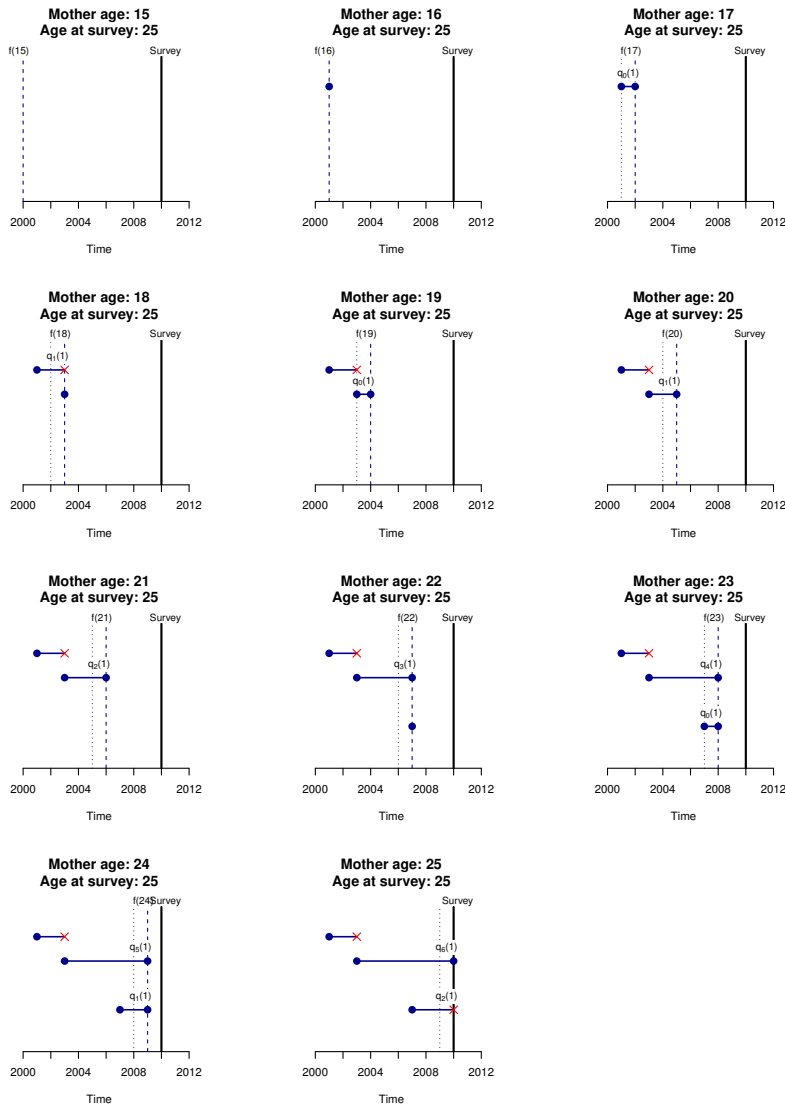
Simulation details and further results

Birth histories for $47 \times 24,000$ women were simulated on a discrete, yearly time scale. For simplicity, we allowed the year prior to the survey to be completely observed and did not allow for births during the survey year, which follows the simple example provided in

Section 2.1. Thus children could be born at any point prior to and including $t_s - 1$ (when the woman was aged $m_s - 1$) and could die in t_s , where t_s is the year of the survey and m_s is the age of the woman at time of survey.

Figure A-1 illustrates how FBH data was simulated for a woman who was 25 at the time of the survey. In the top left panel, when the woman is 15, the probability she gives birth is $f(15)$ (fertility does not change over time). In this example, she does not give birth. In the middle top panel, when the woman is 16, the probability she gives birth is $f(16)$. Here, she does give birth. In the following year (top right panel), the probability the woman gives birth is $f(17)$, and the probability the child dies is ${}_1q_0 = q_0(1)$. As the woman and her children age, we observe her to have three children at ages 16, 18, and 23. One child dies between age 1 and 2 and another dies between age 2 and 3. Her other child survives through the time of the survey. For women with FBH data, this information is completely observed. For women with SBH data, we observe only the total number of children the woman had and the number of those children who died (in this example, three births and two deaths).

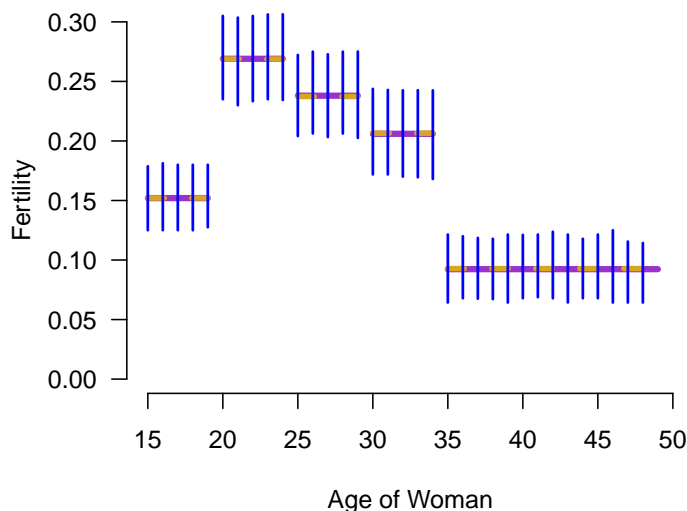
Figure A-1: Illustration of the data-generating mechanism



Note: Suppose a woman is 25 at the time of the survey in 2010 and suppose $f(m) > 0$ for $m \geq 15$. Starting at the top right and proceeding left and down are panels following her and any children she has forward through time starting at age 15. The blue dashed line represents the current year and the black dotted line represents the prior year. Blue circles represent births and survival, red "x"s represent deaths.

Figure A-2 shows the fertilities that were used and the estimates.

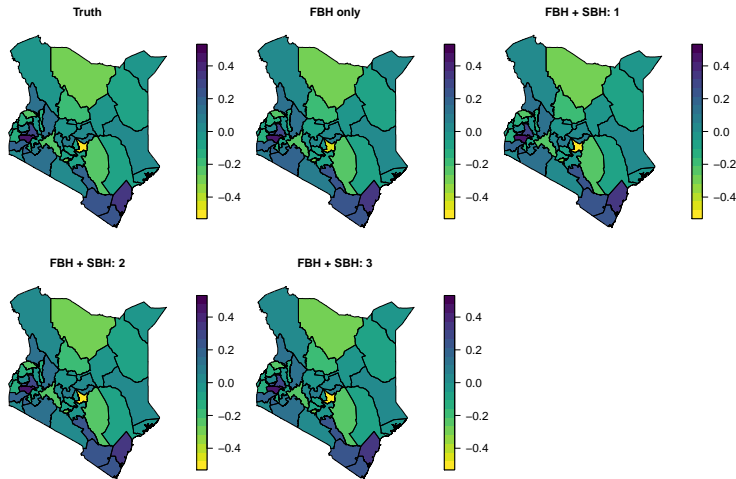
Figure A-2: Simulation study: Fertility estimates under different models



Note: Blue: 5th and 95th percentiles of observed fertility probabilities in the SBH data (by region and surveyed woman's age). Purple: True underlying fertility probabilities. Yellow: Estimated fertility probabilities from FBH data.

The estimates of U5MR, and corresponding measures of uncertainty, by region and time period, and corresponding measures of uncertainty by region and time period were derived using a multivariate normal approximation. That is, defining $\hat{\psi}$ to be the estimates for a generic region and time period and $\hat{\Sigma}$ to be the inverse Hessian obtained from using TMB, we simulate 1,000 draws, from $\hat{\psi}^{(i)} \sim N(\hat{\psi}, \hat{\Sigma})$,

Figure A-3: Simulation study: Mortality structured and unstructured spatial random effect parameter

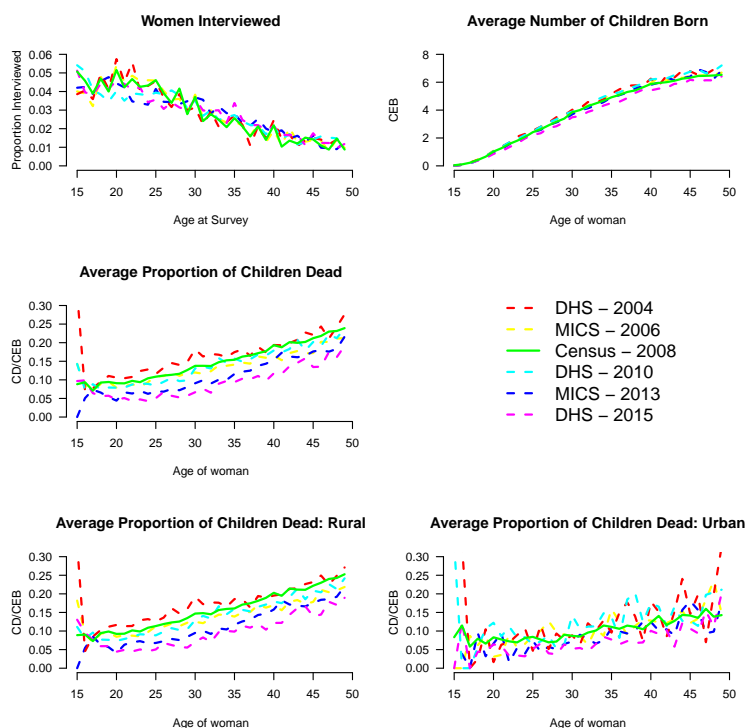


Note: Structured and unstructured spatial random effect parameters, $S_r + \epsilon_{r_i}$, in mortality model. Posterior median estimates are mapped. The truth is compared to FBH analysis and three approaches to dealing with births in the SBH model: (1) true births, (2) true fertilities, (3) estimated fertilities.

Further information for Malawi application

We conducted an exploratory data analysis, comparing the five FBH surveys and census. Results are in Figure A-4. Across all data sources, the age of woman at interview tends to be similar. The reported total number of children born tends to be smaller for more recent surveys, consistent with decreasing fertility over time. This also tends to be the case for the average proportion of children died, consistent with decreasing mortality. However, we do notice that rural women in the census tended to report a higher proportion of children who died than we would expect given the other surveys.

Figure A-4: Application: Exploratory plots



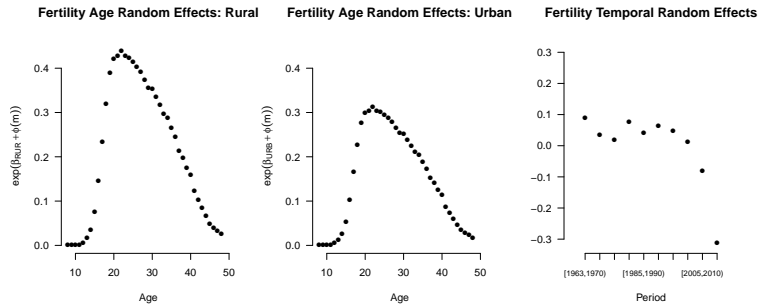
Note: Exploratory plots showing proportion of women interviewed, average number of reported children born, and average proportion of children who died (overall and by strata) by age of woman at the interview. Colors indicate the survey or census year and type.

The parameter estimates for the fertility model are:

- β_{RUR} : -1.97 (-2.01, -1.94)
- β_{URB} : -2.31 (-2.35, -2.28)
- κ_T : 87.1 (25.5, 257)
- κ_A : 0.22 (0.15, 0.33)
- κ_S : 217 (86.6, 630)
- κ_ϵ : 640 (178, 1780)

Figure A-5 shows estimates of the fertility odds (left) and the time trends (right).

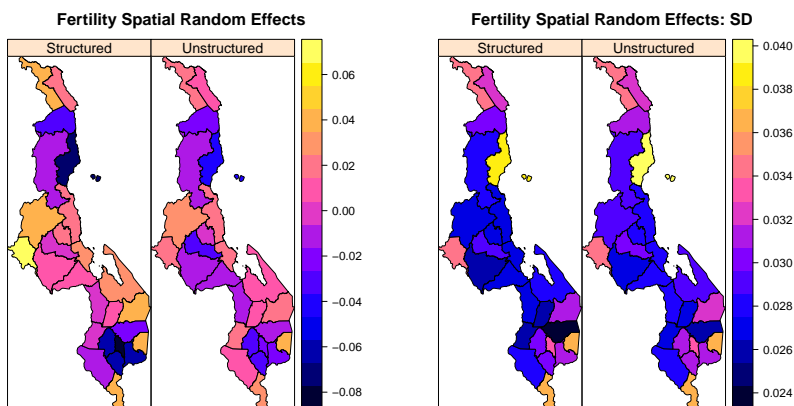
Figure A-5: Application: Estimates of fertility random effects



Note: Left: posterior medians of fertility random effects by strata $\exp(\beta_{\text{strata}}\phi(m))$. Right: posterior medians of time trend $\phi(p)$.

The region-specific spatial and iid adjustments S_r , ϵ_r for fertility are shown in Figure A-6; on the left we give point estimates, and on the right measures of uncertainty. It is clear that the between-region variation is mostly spatially structured, with only a small contribution from the random shocks. The uncertainty in the spatial random effects is greater also.

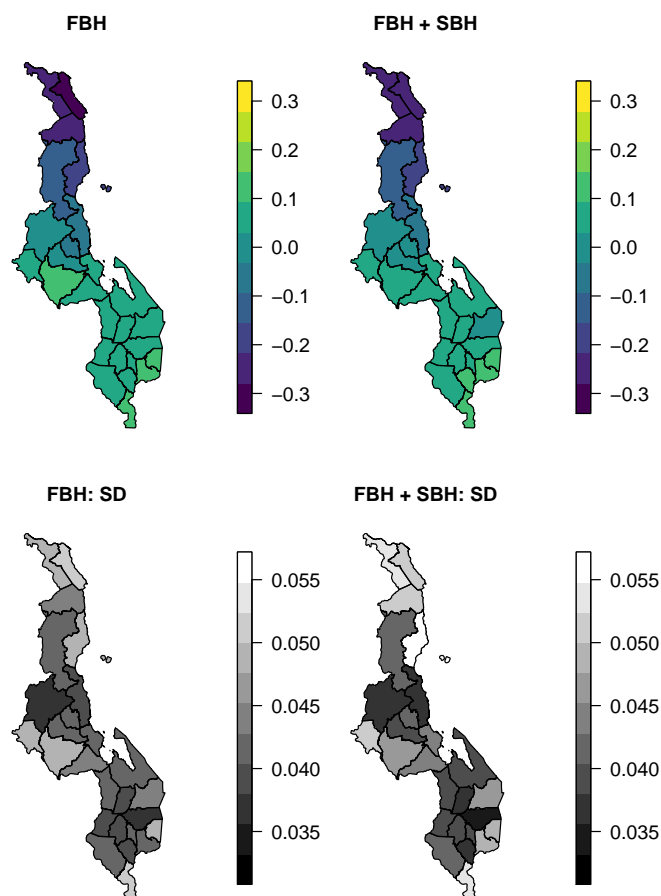
Figure A-6: **Application: Results for the regional structured S_r and unstructured ϵ_r random effects for fertility**



Note: Left: posterior medians. Right: posterior standard deviations.

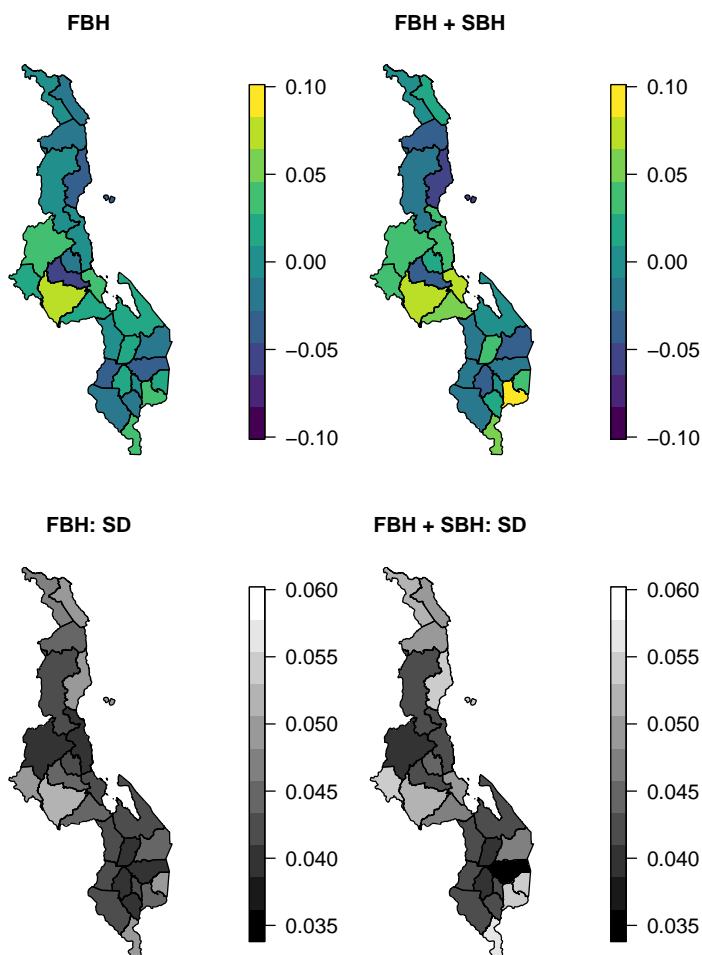
Figures A-7 and A-8 show point estimates of the spatial and iid regional terms, respectively, along with measures of uncertainty, and show that, as with fertility, there is strong spatial structure. Specifically, there is an increasing trend in mortality when moving from north to south.

Figure A-7: Application: Spatial random effect, S_r



Note: Comparison of FBH (left) and FBH + SBH (right) analyses. Top: posterior medians. Bottom: posterior standard deviations.

Figure A-8: **Application: Unstructured random effect, ϵ_r**

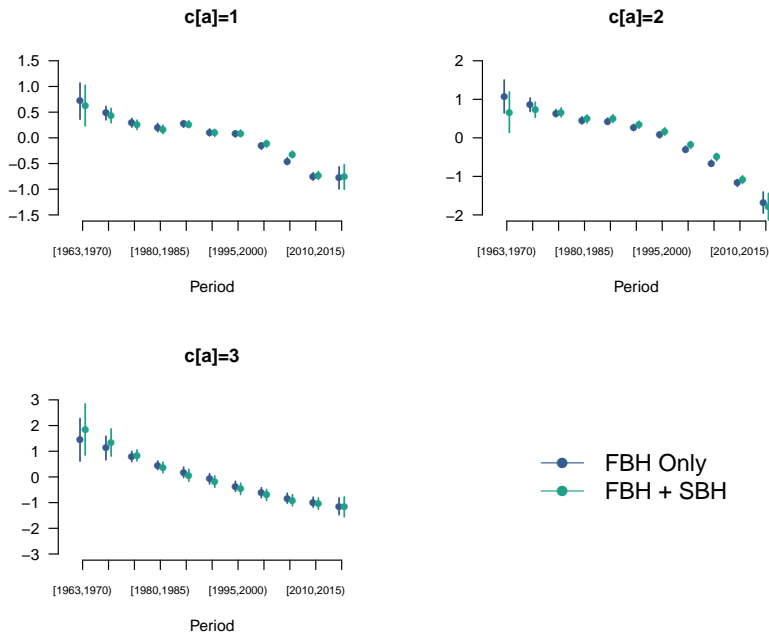


Note: Comparison of FBH (left) and FBH + SBH (right) analyses. Top: posterior medians. Bottom: posterior standard deviations.

Figure A-9 shows the estimates of the age-specific adjustments to the mortality hazards (for three age bands), for the FBH only and for the FBH augmented with SBH data. These are the log hazards for a rural area in a *typical* area, i.e., one with $S_r = \epsilon_r = 0$. There are strong reductions in all three age bands. For ${}_1q_0$ and ${}_1q_1$ there is a flattening

in the 1990s, due to the HIV epidemic. The final (period 11) values are predictions and have wide intervals.

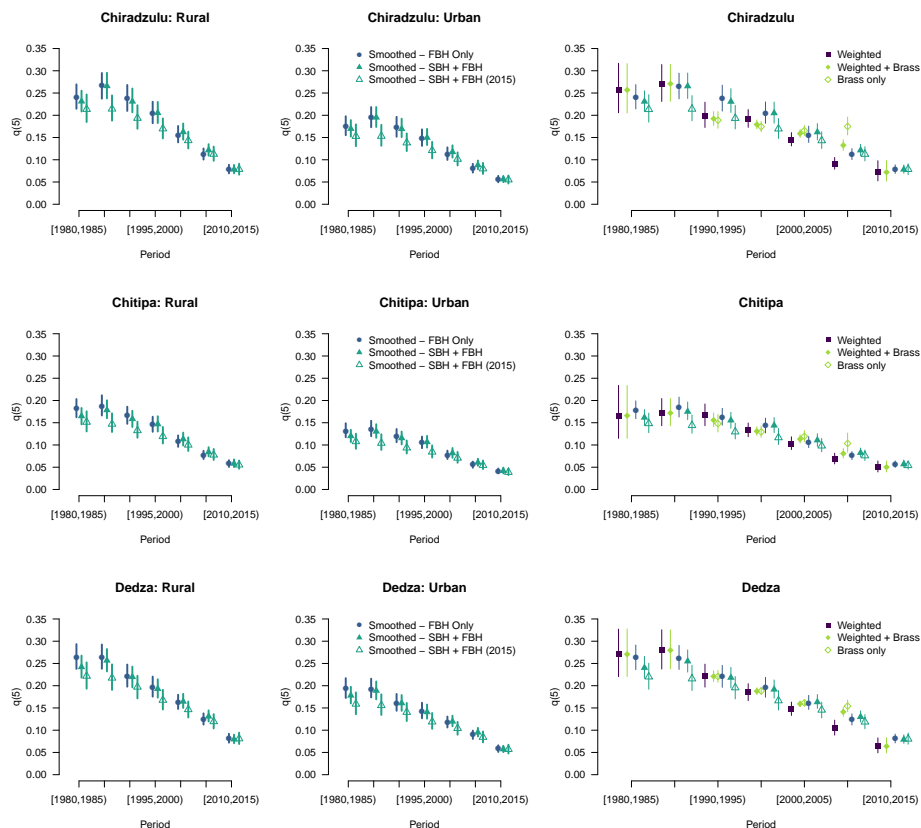
Figure A-9: Application: Age-specific adjustments by period (the RW2 parameters), $\phi_{c[a]}$



Note: Comparison for FBH versus FBH + SBH for three hazards by time period. $c[a] = 1: a = 0$; $c[a] = 2: a = 1, \dots, 4$, $c[a] = 3: a = 5, 6, \dots$

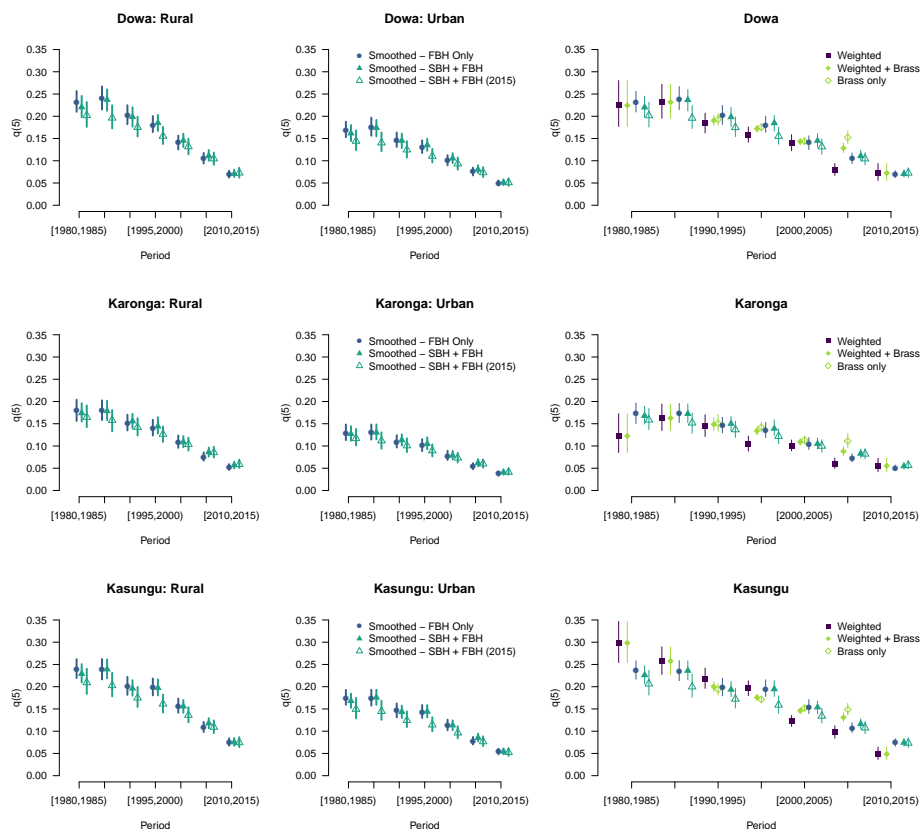
Figures A-10 through A-17 compare posterior medians and 95% uncertainty intervals for 23 regions in Malawi using different methods (the proposed model incorporating only FBH data, our proposed approach to include SBH data [combined with all FBH data and alternatively combined with only the 2015 DHS], direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-10: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural



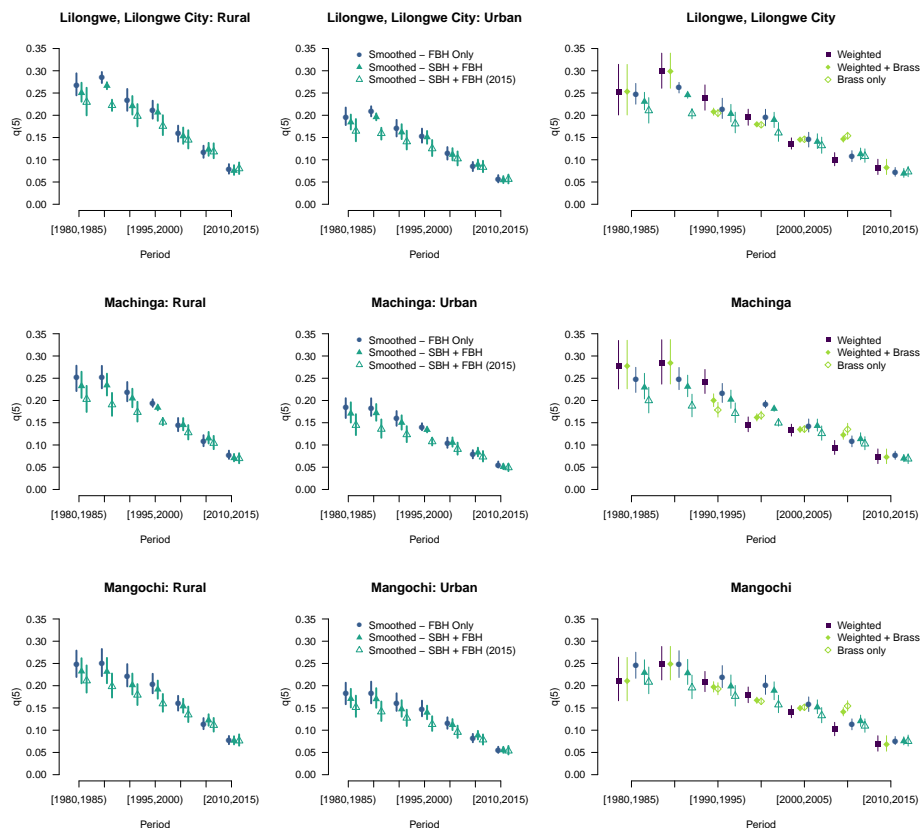
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-11: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 3 regions



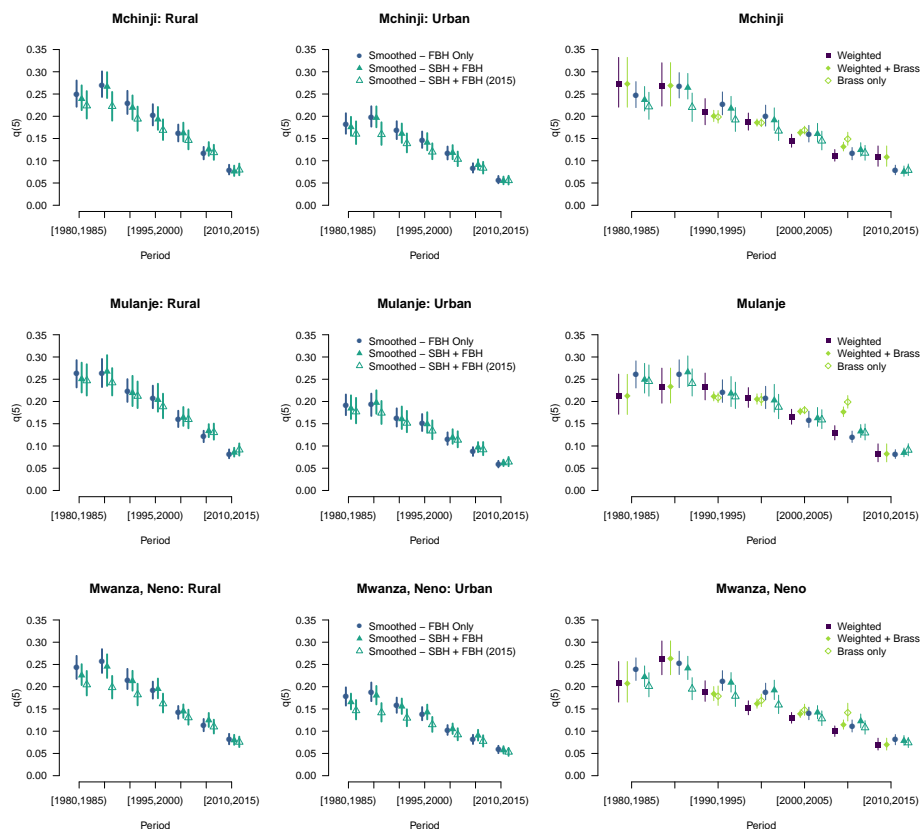
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-12: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 3 regions



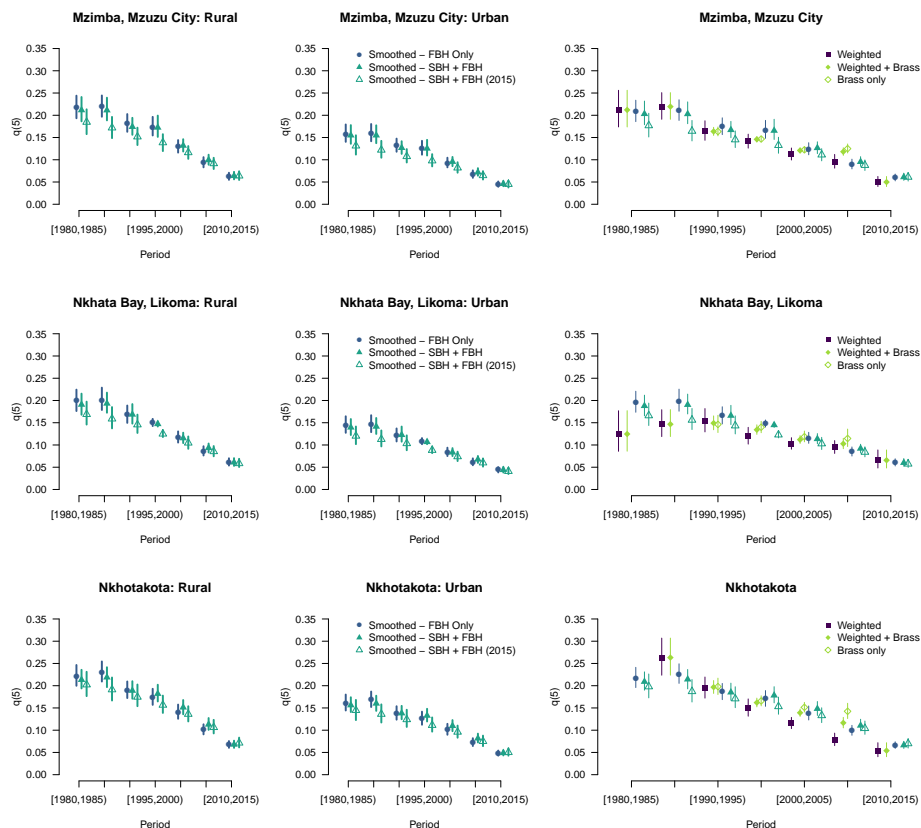
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-13: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 3 regions



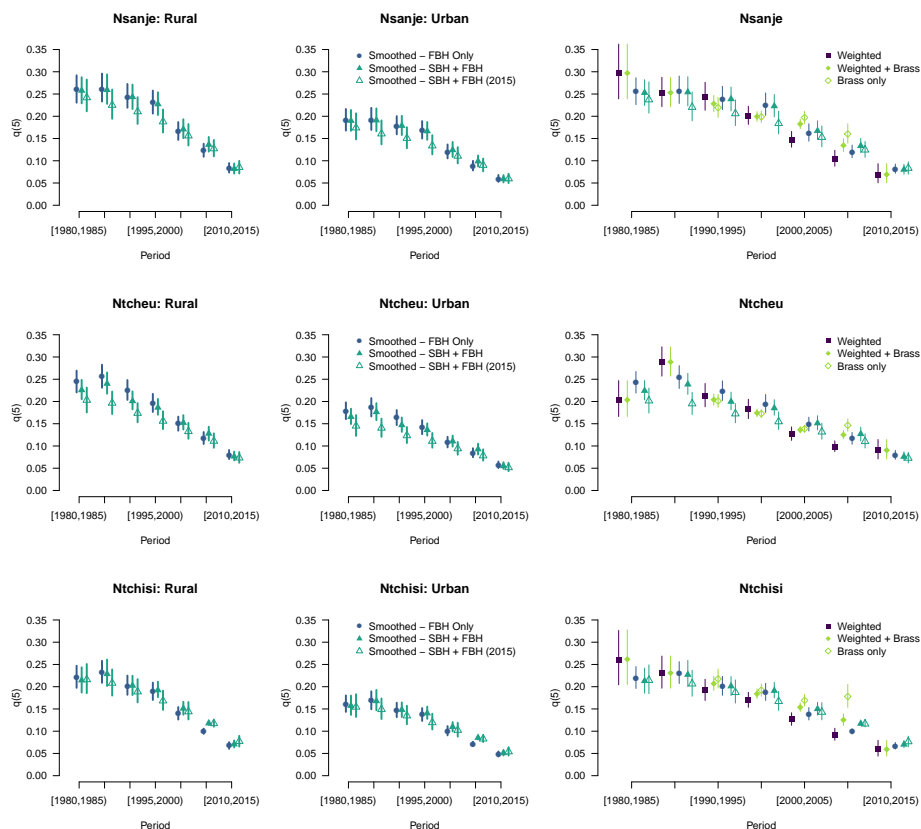
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-14: Application: Posterior medians and 95% uncertainty intervals of USMR, stratified by urban and rural for 3 regions



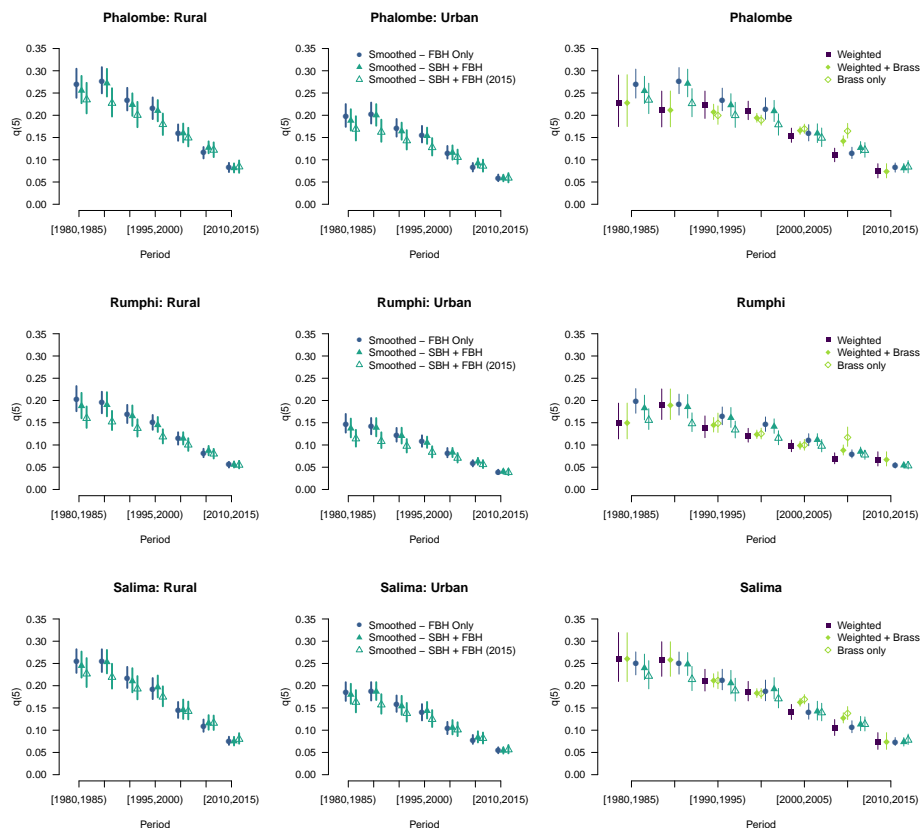
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-15: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 3 regions



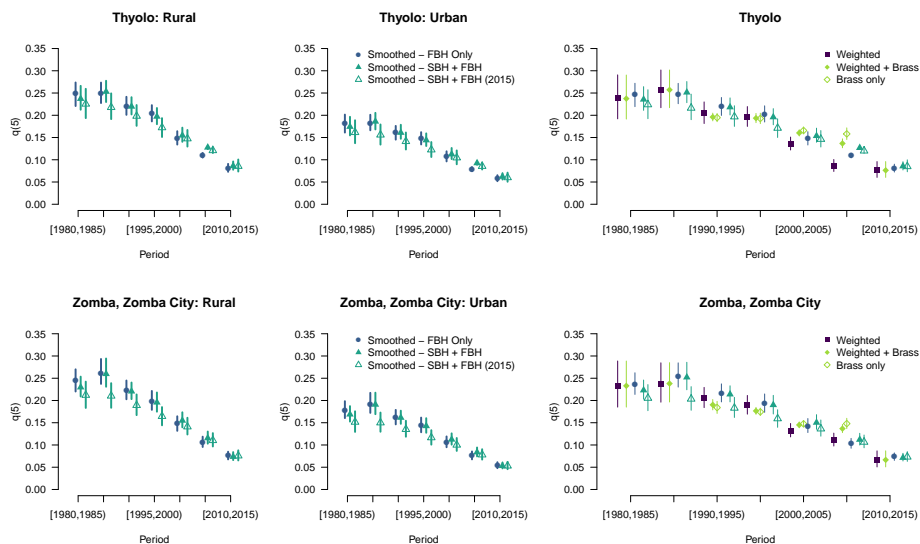
Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Figure A-16: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 3 regions



Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

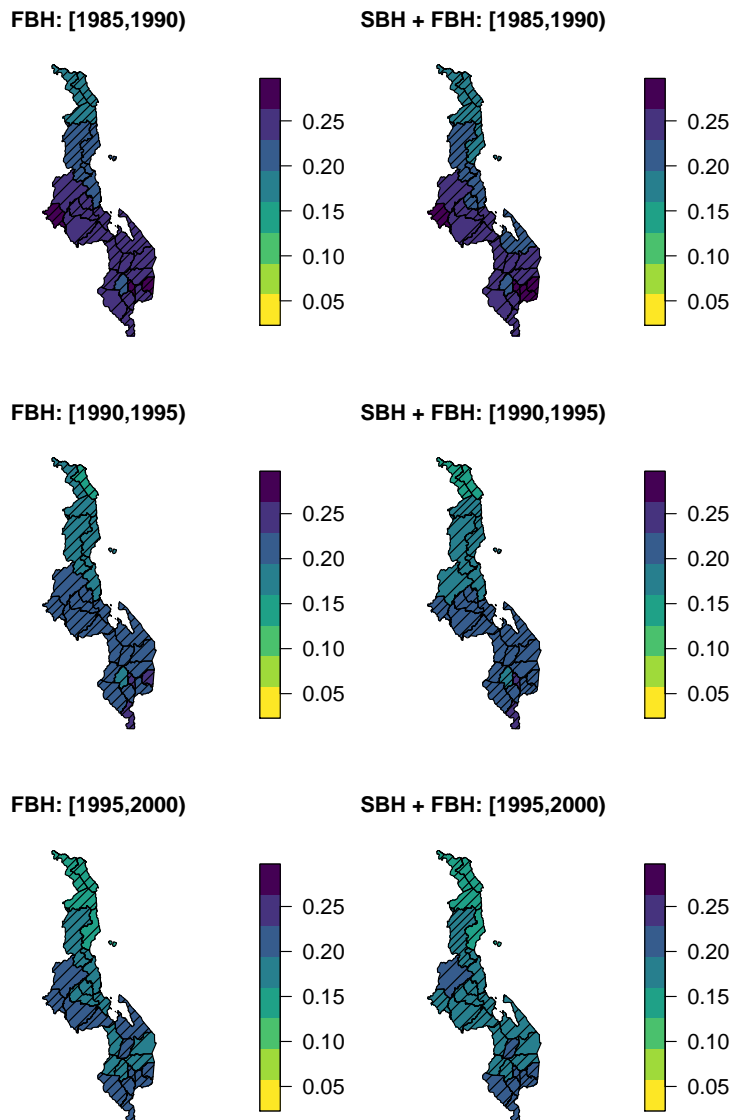
Figure A-17: Application: Posterior medians and 95% uncertainty intervals of U5MR, stratified by urban and rural for 2 regions



Note: Plots on the right are combined over strata and results are compared with direct estimates, direct estimates combined with the Brass method, and the Brass method only.

Maps of the posterior median of U5MR over time are in Figures A-18 and A-19.

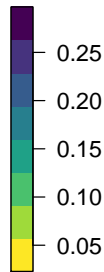
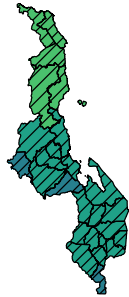
Figure A-18: Application: Maps of U5MR (posterior medians) for 3 time periods



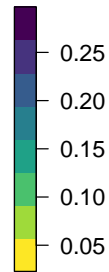
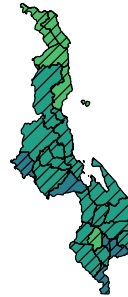
Note: Left: using FBH data only. Right: incorporating SBH data. Denser hatching reflects greater uncertainty (based on the posterior standard deviation of logit U5MR).

Figure A-19: Application: Maps of U5MR (posterior medians) for 2 time periods

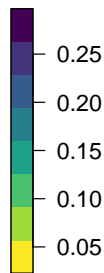
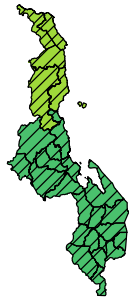
FBH: [2000,2005)



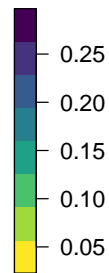
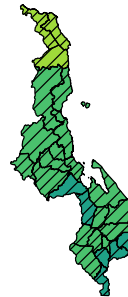
SBH + FBH: [2000,2005)



FBH: [2005,2010)



SBH + FBH: [2005,2010)



□ 0.019 – 0.05	▨ 0.083 – 0.092
▧ 0.05 – 0.058	▩ 0.092 – 0.1
▦ 0.058 – 0.067	▪ 0.1 – 0.12
▤ 0.067 – 0.075	▫ 0.12 – 0.14
▥ 0.075 – 0.083	

Note: Left: using FBH data only. Right: incorporating SBH data. Denser hatching reflects greater uncertainty (based on the posterior standard deviation of logit U5MR).

Computation

This model is implemented in the TMB R package. The TMB package has a less user-friendly interface than R-INLA but has far fewer restrictions on the class of models that can be fit. In particular, R-INLA cannot be used for our model because of the nonlinear mean function. Let \mathbf{z} denote the vector containing $z_a(t)$, \mathbf{d} the vector containing the realized values of $T_D^{m_s}$, \mathbf{B} the vector containing $T_B^{m_s}$, \mathbf{c} the vector containing $c_{m_s}(a, \mathbf{x}(t))$, $\boldsymbol{\beta}$ the vector containing latent mortality fixed effects, and $\boldsymbol{\theta}$ the vector containing latent mortality random effects. In our proposed approach, the negative log posterior is

$$-f(\boldsymbol{\theta}, \boldsymbol{\beta}) = \underbrace{-\log p(\mathbf{z}|\boldsymbol{\beta}, \boldsymbol{\theta})}_{\text{FBH contribution}} - \underbrace{\log p(\mathbf{d}|\mathbf{B}, \mathbf{c}, \boldsymbol{\beta}, \boldsymbol{\theta})}_{\text{SBH contribution}} - \underbrace{\log p(\boldsymbol{\beta}, \boldsymbol{\theta})}_{\text{prior}}.$$

This is specified in a C++ template. After specifying the objective function, the user calls a TMB function that compiles the code and the user can flag any parameters as random effects. That is, $\boldsymbol{\theta}$. TMB uses Laplace approximations to integrate out the random effects. Specifically,

$$L(\boldsymbol{\beta}) = \int \exp[-f(\boldsymbol{\theta}, \boldsymbol{\beta})] d\boldsymbol{\theta} \\ \approx L^*(\boldsymbol{\beta}) = \det\{H(\boldsymbol{\beta})\}^{-1/2} \exp\left[-f(\hat{\boldsymbol{\theta}}(\boldsymbol{\beta}), \boldsymbol{\beta})\right],$$

where $H(\boldsymbol{\beta}) = -\frac{\partial^2}{\partial \boldsymbol{\theta}^2} f(\boldsymbol{\theta}, \boldsymbol{\beta})|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}(\boldsymbol{\beta})}$ and $\hat{\boldsymbol{\theta}}(\boldsymbol{\beta}) = \text{argmin}_{\boldsymbol{\theta}} f(\boldsymbol{\theta}, \boldsymbol{\beta})$. The TMB function returns $-\log L^*(\boldsymbol{\beta})$ and its derivative so that an estimate for $\boldsymbol{\beta}$ can be obtained using nonlinear optimization techniques,

$$\hat{\boldsymbol{\beta}} = \text{argmin}_{\boldsymbol{\beta}} -\log L^*(\boldsymbol{\beta}),$$

and the Hessian can be used to derive an estimate of the uncertainty.

In the TMB algorithm, the parameters (specifically random effects) are unconstrained (define these as $S_r^{(un)}$ and $\phi_{c[a]}(t)^{(un)}$). To obtain the constrained versions (define these as $S_r^{(cons)}$ and $\phi_{c[a]}(t)^{(cons)}$) for use in the likelihood, we use Rue and Held (2005, Equation 2.30):

- First define $Q_j = \kappa_j K_j + 10^{-5}$ for $j = \{T, S\}$ where K is the RW2 “structure matrix.”

- In equation 2.30, $\mathbf{A}_j = [1, \dots, 1]$, where the length of \mathbf{A}_t is the number of periods and \mathbf{A}_s is the number of regions.

The form for the posterior coded up in TMB (up to a constant) for the mortality model in the simulation is therefore:

$$\begin{aligned}
 & \underbrace{-\log p(\mathbf{z}|\beta, \phi^{(cons)}, \mathbf{S}^{(cons)}, \epsilon)}_{\text{FBH contribution}} - \underbrace{-\log p(\mathbf{d}|\mathbf{B}, \mathbf{c}, \beta, \phi^{(cons)}, \mathbf{S}^{(cons)}, \epsilon)}_{\text{SBH contribution}} - \\
 & \quad \underbrace{\log p(\beta)}_{\text{prior for } \beta} + \underbrace{\frac{\kappa_\epsilon}{2} \epsilon^T \epsilon - \frac{47}{2} \log(\kappa_\epsilon)}_{\pi(\epsilon)} + \\
 & \quad \underbrace{\frac{1}{2} \phi^{(un)T} \mathbf{Q}(\kappa_T) \phi^{(un)} - \frac{1}{2} \log |\mathbf{Q}(\kappa_T)|}_{\pi(\phi)} + \\
 & \quad \underbrace{\frac{1}{2} \mathbf{S}^{(un)T} \mathbf{Q}(\kappa_S) \mathbf{S}^{(un)} - \frac{1}{2} \log |\mathbf{Q}(\kappa_S)|}_{\pi(S)} - \\
 & \quad \underbrace{\log p(\kappa_T)}_{\text{prior for } \kappa_T} - \underbrace{\log p(\kappa_S)}_{\text{prior for } \kappa_S} - \underbrace{\log p(\kappa_\epsilon)}_{\text{prior for } \kappa_\epsilon}.
 \end{aligned}$$

Prior specifications

Some care is in general required when specifying prior distributions, particularly for variance components. A very appealing and rigorous approach has been developed recently (Simpson et al. 2017) under the name *penalized complexity (PC)* priors. The basic idea is to take a baseline (simple) model and then penalize departures from this model. For example, it is well-known that there can be sensitivity to the prior on the variance when random effects models are fitted to data. Under the PC model, the baseline model corresponds to a variance of zero (in which case all random effects are zero), and greater values of the random effects standard deviation are more and more penalized (i.e., discouraged in the prior). To specify these priors, one sets two values for each parameter, a value of the parameter (on an interpretable scale), below we call this u , and a prior probability of exceedance of this value, which we call α .

For the mortality model in the simulation we use the following as (independent)

priors:

$$\begin{aligned}\beta_c &\sim N(0, 10^2), & \kappa_T &\sim \text{PCprior}(u = 1, \alpha = 0.01), \\ \kappa_S &\sim \text{PCprior}(u = 1, \alpha = 0.01), & \kappa_\epsilon &\sim \Gamma(\text{shape} = 1, \text{scale} = 200).\end{aligned}$$

For our Malawi model, we specify the following priors. For fertility:

- $\kappa_T \sim \text{PCPrior}(u = 0.5, \alpha = 0.01)$ (precision for $\phi_{c[m]}(p)$, though u is on the standard deviation scale)
- $\kappa_S \sim \text{PCPrior}(u = 0.5, \alpha = 0.01)$ (precision for S_r , though u is on the standard deviation scale)
- $\kappa_\epsilon \sim \Gamma(1, 1/400)$ (precision for ϵ_r)
- $\beta_X \sim N(0, 100)$

For mortality:

- $\kappa_T \sim \text{PCPrior}(u = 1, \alpha = 0.01)$ (precision for $\phi_{c[a]}(p)$)
- $\kappa_S \sim \text{PCPrior}(u = 1, \alpha = 0.01)$ (precision for S_r)
- $\kappa_\epsilon \sim \Gamma(1, 1/200)$ (precision for ϵ_r)
- $\beta_X \sim N(0, 100)$
- $\beta_{\text{SBH}}, \beta_{\text{SBH,URB}} \sim N(0, 10)$