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Formal Relationship

On a closed-form expression and its approximation to Gompertz life disparity

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Abstract

BACKGROUND
In the literature, there exists a closed form solution to the remaining life expectancy at age x when mortality is governed by the Gompertz law. This expression contains a special function that allows us to construct high-accuracy approximations, which are also helpful in assessing the elasticity of life expectancy with respect to the model parameters. However, to my knowledge, a similar formulation for life disparity does not exist, and as a consequence, it does not exist for life table entropy either.

CONTRIBUTION
Under the assumption that mortality is governed by the Gompertz law, I present and prove a closed form expression for life disparity at age x that is similar to that existing for life expectancy. Since the closed form expressions hold for both life expectancy and life disparity, an exact expression for the life table entropy is immediately derived. In addition, using known relationships on the exponential integral function, an approximate form for life disparity is also obtained.

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1. Relationship

Let x be any initial age, with $x \geq 0$. The life table survival function, denoted by $l(x)$, provides the survival probability from birth to age x. It is related to the distribution of deaths, denoted by $f(x)$, through

$$f(x) = -\frac{\mathrm{d}l(x)}{\mathrm{d}x}, \qquad l(x) = \int_x^\infty f(s)\, ds = 1 - \int_0^x f(s)\, ds.$$

The force of mortality, denoted by $\mu(x)$, is related to the previous functions by

$$\mu(x) = \frac{f(x)}{l(x)} \quad \text{for all } x \text{ such that } l(x) > 0.$$

Let

$$e(x) = \frac{\int_x^\infty l(s)\, ds}{l(x)}$$

express the life expectancy and

$$e^\dagger(x) = \frac{\int_x^\infty e(s) f(s)\, ds}{l(x)} \tag{1}$$

express the life disparity, both of them at age x. Starting from these relationships, as in Vaupel and Romo (2003) and Aburto et al. (2019), the life table entropy above age x is proven to be equal to

$$\overline{H}(x) = \frac{e^\dagger(x)}{e(x)}.$$

When the Gompertz model is used to represent mortality, it is

$$\mu(x) = a e^{kx} \quad x \geq 0,$$

where a and k denote the initial mortality level and the rate of aging, respectively, with $a > 0$ and $k > 0$. As shown, for example, in Pollard (2002), Pflaumer (2011), and Castellares et al. (2020), the closed form expression for the remaining life expectancy at age x is provided by

$$e(x) = \frac{1}{k}\exp\Big(\frac{\mu(x)}{k}\Big)E_1\Big(\frac{\mu(x)}{k}\Big), \tag{2}$$

where $E_1(z) = \int_1^\infty t^{-1}\exp(-zt)\,dt = \int_z^\infty t^{-1}\exp(-t)\,dt$ is the exponential integral function defined for $z \in \mathbb{C}$.

Starting from (2), all the relationships developed throughout the paper assume that the Gompertz law is used to represent mortality, and no special notation is adopted.

I prove that an analogous expression can be obtained for life disparity at age x. That is, the following relationship holds:

$$e^\dagger(x) = \frac{1}{k}\exp\Big(\frac{\mu(x)}{k}\Big)E_2\Big(\frac{\mu(x)}{k}\Big), \tag{3}$$

where $E_2(z) = \int_1^\infty t^{-2}\exp(-zt)\,dt = \int_z^\infty t^{-2}\exp(-t)\,dt$ is the exponential integral function of order 2 defined for $z \in \mathbb{C}$.

As a consequence, from the definition of the life table entropy using (2) and (3), it immediately follows that

$$\overline{H}(x) = \frac{E_2\Big(\frac{\mu(x)}{k}\Big)}{E_1\Big(\frac{\mu(x)}{k}\Big)}. \tag{4}$$

Note that closed form expressions (3) and (4) assume that mortality is consistent with the Gompertz law, which is empirically valid over adult ages across most countries around the world despite the known limitations at both young and very advanced ages. In this last regard, some studies suggest that the exponential growth of mortality with age is followed by a period of deceleration, with slower rates of mortality increase. See Horiuchi and Wilmoth (1998) and Thatcher, Kannisto, and Vaupel (1998), among many others. Therefore mortality at advanced ages does not seem properly modeled by the Gompertz law, as illustrated in Gavrilov and Gavrilova (1991) or more recently Barbi et al. (2018). This point is currently subject to some debate in the demographic literature. Gavrilov and Gavrilova (2011) refused the assumption of the mortality deceleration and illustrated that the Gompertz model provides a better fit of mortality data than the logistic models in the age range from 80 to 106. Gavrilova, Gavrilov, and Krut'ko (2017) found that mortality after age 110 years continues to grow with age and can be fitted by the Gompertz law. However, this issue does not affect the demonstration of relationships (3) and (4), although the goodness of the fit on data of the Gompertz law suggests using them for adult ages.

2. Proof

Let $\mu(x) = ae^{kx}$ be the Gompertz law for the force of mortality. Consequently, the survival function of the Gompertz model is $l(x) = \exp(\frac{a}{k})\exp\Big(-\frac{\mu(x)}{k}\Big)$. See e.g. Castellares et al. (2020).

First, I have considered the term inside the integral in (1), for which it is

$$e(x)f(x) = e(x)\mu(x)l(x) = \exp\Big(\frac{a}{k}\Big)\frac{\mu(x)}{k}E_1\Big(\frac{\mu(x)}{k}\Big),$$

so that

$$\begin{aligned}\int_x^\infty e(s)f(s)\,ds &= \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_x^\infty \mu(s)E_1\Big(\frac{\mu(s)}{k}\Big)\,ds = \\ &= \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_x^\infty \mu(s)\Big(\int_1^\infty \frac{\exp(-\frac{\mu(s)}{k}t)}{t}\,dt\Big)\,ds.\end{aligned}$$

Using the reversal of the integration, it follows that

$$\int_x^\infty e(s)f(s)\,ds = \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_1^\infty \frac{1}{t}\Big(\int_x^\infty \mu(s)\exp\big(-\frac{\mu(s)}{k}t\big)\,ds\Big)\,dt,$$

and by means of the substitution $y = \mu(s)$, it is

$$\begin{aligned}\int_x^\infty e(s)f(s)\,ds &= \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_1^\infty \frac{1}{t}\Big(\int_{\mu(x)}^\infty \frac{1}{k}\exp\big(-\frac{y}{k}t\big)\,dy\Big)\,dt = \\ &= \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_1^\infty \frac{1}{t}\Big(\frac{1}{k}\exp\big(-\frac{y}{k}t\big)\frac{-k}{t}\Big|_{\mu(x)}^\infty\Big)\,dt = \\ &= \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\int_1^\infty \frac{\exp\big(-\frac{\mu(x)}{k}t\big)}{t^2}\,dt = \frac{1}{k}\exp\Big(\frac{a}{k}\Big)E_2\Big(\frac{\mu(x)}{k}\Big).\end{aligned}$$

Finally, dividing by $l(x)$, relationship (3) follows, and as a consequence, relationship (4) is also derived.

Q.E.D.

3. History and related results

Wrycza (2014) finds that the life table entropy at age 0 is provided by $\overline{H} = \frac{1}{b}\Big(\frac{1}{e_0} - a\Big)$, where $\overline{H}$ and e_0 mean $\overline{H}(0)$ and $e(0)$, respectively, when the force of mortality follows the Gompertz law. This is a special case of (4). Indeed, as shown in Abramowitz and Stegun (1964), the following recurrence relation holds:

$$E_{n+1}(z) = \frac{1}{n}[\exp(-z) - zE_n(z)] \qquad n = 1, 2, 3, ...$$

Then for $n = 1$, and set $z = \frac{\mu(x)}{k}$, the following relationship holds:

$$E_2\Big(\frac{\mu(x)}{k}\Big) = \exp\Big(-\frac{\mu(x)}{k}\Big) - \frac{\mu(x)}{k}E_1\Big(\frac{\mu(x)}{k}\Big).$$

Hence, substituting the previous relationship in (3), it follows that

$$\begin{aligned} e^\dagger(x) &= \frac{1}{k}\exp\Big(\frac{\mu(x)}{k}\Big)\Big[\exp\Big(-\frac{\mu(x)}{k}\Big) - \frac{\mu(x)}{k}E_1\Big(\frac{\mu(x)}{k}\Big)\Big] = \\ &= \frac{1}{k}\Big[1 - \exp\Big(\frac{\mu(x)}{k}\Big)\frac{\mu(x)}{k}E_1\Big(\frac{\mu(x)}{k}\Big)\Big], \end{aligned}$$

and from (2), it immediately follows that

$$e^\dagger(x) = \frac{1}{k}\big[1 - \mu(x)e(x)\big]. \tag{5}$$

Hence, substituting (5) in (4), it is

$$\overline{H}(x) = \frac{1}{k}\Big(\frac{1}{e(x)} - \mu(x)\Big), \tag{6}$$

as in Aburto et al. (2019), and when $x = 0$ the result in Wrycza (2014) is obtained.

In addition, note that the expression in the square brackets on the right-hand side in (5) is the derivative, with the negative sign, of life expectancy with respect to x. See Finkelstein and Vaupel (2009). Therefore, relationship (5) can be written as

(7) $$e^{\dagger}(x) = -\frac{1}{k}\frac{\mathrm{d}e(x)}{\mathrm{d}x}.$$

This means that the average number of life-years lost as a result of death at age x can be approximately given by the change in life expectancy between ages x and $x+1$ over the rate of aging. For example, let us consider the life table for Italian females in 2007, as downloaded from the Human Mortality Database (accessed in November 2022). On this data, for all ages in the table, fitting the Gompertz model with the method of least squares, the parameters that best fit the data are $a = 3,12002 \times 10^{-5}$ and $k = 0,088799043$. For Italian females, for the year 2007, life expectancy at birth and at age one are $e(0) = 84.04$ and $e(1) = 83.3$ years, respectively. It follows that the life disparity at birth for Italian females for the year 2007 is about 8.4 years. (In this regard, see Vaupel, Zhang, and van Raalte 2011, where the corresponding value, calculated for the same population and the same year, is 8.8 years.)

Furthermore, using (7), it immediately follows that

(8) $$\overline{H}(x) = -\frac{1}{k}\acute{e}(x),$$

where $\acute{e}(x)$ denotes the relative derivative with respect to x of life expectancy, namely $\acute{e}(x) = \dfrac{\frac{\mathrm{d}e(x)}{\mathrm{d}x}}{e(x)}$. According to this relationship, when the Gompertz model is used, the life table entropy at age x is proportional (with the negative sign) to the ratio of the relative change in life expectancy and the relative change in mortality at that age, where the latter equals the rate of aging.

In the literature, there exists an approximation formula for Gompertz life expectancy. See Pflaumer (2011) and Missov and Lenart (2013), although in the latter, the procedure described is applied to a wrongly written formula for life expectancy. The approximation formula is derived from (2) using the series expansion of function $E_1(z)$. See Abramowitz and Stegun (1964). By truncating the series after the linear term, it follows that

$$e(x) \approx \frac{1}{k}\exp\Big(\frac{\mu(x)}{k}\Big)\Big(-\gamma - \ln\frac{a}{k} - kx + \frac{\mu(x)}{k}\Big),$$

where $\gamma \approx 0.57722\ldots$ is the Euler-Mascheroni constant.

When $x = 0$ and under the assumption $\frac{a}{k} \approx 0$, as for the human population, the approximation formula for life expectancy at birth is

$$e(0) \approx \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\Big(-\gamma - \ln\frac{a}{k} + \frac{a}{k}\Big) \approx \frac{1}{k}\Big(-\gamma - \ln\frac{a}{k}\Big).$$

Taking into account that $m = \frac{1}{k}\ln\frac{k}{a}$, where m denotes the modal age at death (see Pollard and Valkovics 1992 and Pflaumer 2011, among many others), the well-known relationship also follows:

$$e(0) \approx m - \frac{\gamma}{k}.$$

Analogously, let us consider the series expansion to function $E_2(z)$ − see Abramowitz and Stegun (1964) − which is

$$\begin{aligned} E_2(z) &= -z[-\ln z - \gamma + 1] - \sum_{m=0, m\neq 1}^{\infty} \frac{(-z)^m}{(m-1)\cdot m!} \\ &= -z[-\ln z - \gamma + 1] - \big(-1 + \frac{z^2}{2} - \frac{z^3}{12} + \ldots\big). \end{aligned}$$

By truncating the series after the first term, it follows that

$$E_2(z) \approx 1 - z[-\ln z - \gamma + 1],$$

and, substituting it in formula (3), then it follows that

$$e^{\dagger}(x) \approx \frac{1}{k}\exp\Big(\frac{\mu(x)}{k}\Big)\Big(1 - \frac{\mu(x)}{k}\Big[-\ln\frac{a}{k} - kx - \gamma + 1\Big]\Big). \tag{9}$$

When $x = 0$ and under the assumption $\frac{a}{k} \approx 0$, it follows that

$$e^{\dagger}(0) \approx \frac{1}{k}\exp\Big(\frac{a}{k}\Big)\Big(1 - \frac{a}{k}\Big[-\ln\frac{a}{k} - \gamma + 1\Big]\Big) \approx \frac{1}{k}\Big(1 - am\Big), \tag{10}$$

which, when $a \approx 0$, provides approximation $e^{\dagger}(0) \approx \frac{1}{k}$, as in Vaupel and Romo (2003).

Furthermore, an approximation formula can also be derived for the life table entropy.

Note that the following relationship holds:

$$E_n(y) = y^{n-1}\Gamma(1-n, y) \quad n = 0, 1, 2, ..., \quad y > 0.$$

See Abramowitz and Stegun (1964). Hence, for $n = 1$ and $n = 2$, we have

$$E_1(y) = \Gamma(0, y) \qquad E_2(y) = y\Gamma(-1, y), \tag{11}$$

respectively. As proven in Jameson (2016) - see Theorem 1, Formula 13 - the following recursive relationship holds for $y > 0$ and all a:

$$\Gamma(a+1, y) = a\Gamma(a, y) + y^a e^{-y}. \tag{12}$$

Using (12) for $a = -1$, from the second relationship in (11), it follows that

$$E_2(y) = \frac{1}{\exp(y)} - y\Gamma(0, y) = \frac{1}{\exp(y)} - yE_1(y),$$

and hence from (4), set $y = \frac{\mu(x)}{k}$, it follows that

$$\overline{H}(x) = \frac{1}{\exp(\frac{\mu(x)}{k})E_1\left(\frac{\mu(x)}{k}\right)} - \frac{\mu(x)}{k}. \tag{13}$$

From (13), when $x = 0$, it is

$$\overline{H}(0) = \frac{1}{\exp(\frac{a}{k})E_1\left(\frac{a}{k}\right)} - \frac{a}{k}, \tag{14}$$

and, again, under assumption $\frac{a}{k} \approx 0$, it follows that

$$\overline{H}(0) \approx \frac{1}{E_1\left(\frac{a}{k}\right)} \approx -\frac{1}{\gamma + \log \frac{a}{k}}. \tag{15}$$

See Pflaumer (2011).

4. Conclusions

I developed an exact formula for determining life disparity at any age x when a Gompertz law is used to model mortality. As an immediate consequence, to express life table entropy, $\overline{H}(x)$, at any fixed age x, a corresponding exact formula, which relies solely on the numerical evaluation of the exponential integral functions and the parameters of the model, is obtained. An approximation formula for life disparity is also obtained based on the series expansions of the integral exponential functions.

Starting from the basic formulas introduced in this paper, further developments aim at deriving closed form expressions for the elasticity of the life table entropy depending on the parameters of the Gompertz law. As the proven relationships can be helpful in other fields such as biology, actuarial studies, and financial engineering, other possible results will explore how to obtain closed form expressions in the case of the present values of life annuities under a Gompertz law mortality assumption.

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