DEMOGRAPHIC RESEARCH
A peer-reviewed, open-access journal of population sciences

DEMOGRAPHIC RESEARCH

VOLUME 50, ARTICLE 44, PAGES 1281–1300
PUBLISHED 12 JUNE 2024
https://www.demographic-research.org/Volumes/Vol50/44/
DOI: 10.4054/DemRes.2024.50.44

Formal Relationship

The Average Uneven Mortality index: Building on the 'e-dagger' measure of lifespan inequality

Marco Bonetti

Ugofilippo Basellini

Andrea Nigri

This publication is part of the Special Collection in the Memory of Professor James W Vaupel (1945–2022), founder and long-time publisher of Demographic Research. The Special Collection is edited by Jakub Bijak, Griffith Feeney, Nico Keilman, and Carl Schmertmann.

Contents

Demographic Research: Volume 50, Article 44
Formal Relationship

The Average Uneven Mortality index: Building on the 'e-dagger' measure of lifespan inequality

Marco Bonetti[1]
Ugofilippo Basellini[2]
Andrea Nigri[3]

Abstract

BACKGROUND
In recent years, lifespan inequality has become an important indicator of population health. Uncovering the statistical properties of lifespan inequality measures can provide novel insights on the study of mortality.

METHODS
We introduce the 'Average Uneven Mortality' (AUM) index, a novel mortality indicator for the study of mortality patterns and lifespan inequality. We prove some new properties of interest, as well as relationships with the 'e-dagger' and entropy measures of lifespan inequality.

RESULTS
The use of the AUM index is illustrated through an application to observed period and cohort death rates from the Human Mortality Database. We explore the behavior of the index across age and over time, and we study its relationship with life expectancy. The AUM index at birth declined over time until the 1950s, when it reverted its trend; also, the index generally increases with age.

CONTRIBUTION
The AUM index is a normalized version of Vaupel and Canudas-Romo's e-dagger measure that can be meaningfully compared across countries and over time. Additionally, we derive an upper bound for both e-dagger and the life-table entropy measures, which are novel formal results. Finally, we develop novel routines to compute e-dagger and the standard deviation of lifetimes from death rates, which are often more precise than available software, particularly for calculations involving older ages.

[1] Carlo F. Dondena Research Center, Bocconi University, Milan, Italy. Email: marco.bonetti@unibocconi.it.
[2] Max Planck Institute for Demographic Research, Rostock, Germany.
[3] University of Foggia, Foggia, Italy.

1. Relationship

1.1 The AUM index, a novel mortality indicator

Let T be a nonnegative random variable denoting time to death, which is distributed according to the probability density function $f(t)$. Let $S(t) = \exp[-\int_0^t \lambda(u)du] = \exp[-\Lambda(t)]$ denote the survival function, that is the fraction of the population expected to survive at least t years, where $\lambda(t)$ indicates the force of mortality or hazard function of the population, and $\Lambda(t)$ the cumulative hazard function of T.

We define the 'Average Uneven Mortality' (AUM) index as the correlation coefficient between the time to death random variable T and its transformation through its own cumulative hazard function:

$$\mathrm{AUM} = \mathrm{Corr}[T, \Lambda(T)] \quad \text{with } \; 0 < \mathrm{AUM} \leq 1\,. \tag{1}$$

The AUM index can be used to study mortality patterns and lifespan inequality. It ranges between 0 and 1, and it is equal to 1 if and only if T has an exponential distribution with parameter θ ($T \sim Exp(\theta)$; for a proof of this result, see Section 2). As a consequence, the index can help determine whether the hazard rate is constant (AUM = 1) or varies with age (AUM < 1). Note that this can be particularly useful when one considers the tail of a survival distribution – that is, the distribution of T conditionally on surviving up to some age a.

1.2 An upper bound for $e^\dagger$ and for $\mathcal{H}$

Let $e^\dagger = \int_0^\omega e(u)f(u)du$ denote the life lost to mortality (Vaupel and Canudas-Romo 2003), where the remaining life expectancy at age u is $e(u) = \int_u^\omega S(t)dt/S(u)$, and ω is the highest age attained in the population. Further, let σ_T denote the standard deviation of the time to death T, $\sigma_T = \left[\int_0^\omega (x - e_0)^2 f(x)dx\right]^{1/2}$, where for brevity we use the shorthand notation $e_0 = e(0)$.

The AUM index defined above can also be written as

$$\mathrm{AUM} = \frac{e^\dagger}{\sigma_T}$$

so that it can also be interpreted as a normalized version of the $e^\dagger$ inequality measure; as such, the index allows one to conduct lifespan inequality comparisons across countries and over time due to its fixed $(0, 1]$ support.

The support of the AUM index implies an upper bound on $e^\dagger$. In particular, it holds that:

$$0 < e^\dagger \leq \sigma_T \, ; \tag{2}$$

that is, $e^\dagger$ is bounded from above by the standard deviation of the time to death T.

Let $\overset{\bullet}{\mathrm{AUM}}$ denote the partial derivative of AUM with respect to the calendar year.[4] In Section 2 we also show that the relative derivative of AUM is such that

$$\log(\overset{\bullet}{\mathrm{AUM}}) \lesseqgtr 0 \quad \Longleftrightarrow \quad \log(\overset{\bullet}{e^\dagger}) \lesseqgtr \log(\overset{\bullet}{\sigma_T}) \tag{3}$$

that is, the relative change of the AUM index is positive (negative) when the relative change in $e^\dagger$ is greater (lower) than the relative change in σ_T.

Lastly, one may also consider the relationship between the AUM index and the entropy measure $\mathcal{H} = e^\dagger/e_0$ (Leser 1955; Keyfitz 1977; Demetrius 1974). The entropy can be expressed in terms of the AUM index as $\mathcal{H} = AUM \cdot CV$, where $CV = \sigma_T/e_0$ is the coefficient of variation of the distribution. From this, one obtains the additional interesting result:

$$0 < \mathcal{H} \leq CV. \tag{4}$$

Note that all these results also hold more generally when the indices are computed conditionally on surviving until any given age a.

2. Proof

We start by proving that the support of the AUM index is $(0, 1]$, as stated in Equation (1). Below we note that AUM > 0 holds; AUM ≤ 1 holds by the correlation inequality. We now prove that AUM is equal to 1 if and only if T has an exponential distribution. On the one hand, if $T \sim Exp(\theta)$, then $\Lambda(t) = \theta t$ and $\mathrm{Corr}[T, \Lambda(T)] = \mathrm{AUM} = 1$. On the other hand, if $\mathrm{AUM} = \mathrm{Corr}[T, \Lambda(T)] = 1$, then $\Lambda(t) = a + bt$. Since $\Lambda(0) = 0$, then $a = 0$, implying that $\Lambda(t) = bt$ and $S(t) = e^{-bt}$ – that is, $T \sim Exp(b)$.

We now show that $0 < e^\dagger \leq \sigma_T$. Schmertmann (2020) shows that Vaupel and Canudas-Romo's $e^\dagger$ equals

$$e^\dagger = \mathrm{Cov}[T, \Lambda(T)] \tag{5}$$

[4] In the following, a dot over a function will denote its partial derivative with respect to calendar year y (which may refer to either a given time period or birth cohort), but we drop the notation y to ease readability.

(for an alternative proof of that relationship, see Supplementary Materials).

Rewriting (5) in terms of correlation rather than covariance produces

$$e^{\dagger} = \mathrm{Corr}[T, \Lambda(T)] \cdot \sigma_T \cdot \sigma_{\Lambda(T)}.$$

We consider the well-known fact that, for any hazard function $\lambda(t)$, $\Lambda(T) \sim Exp(1)$ (for a short proof, see Supplementary Materials). This implies that $\sigma_{\Lambda(T)} = 1$, yielding the interesting new expression

$$e^{\dagger} = \mathrm{Corr}[T, \Lambda(T)] \cdot \sigma_T.$$

Given the definition $\mathrm{AUM} = \mathrm{Corr}[T, \Lambda(T)]$, we have

$$\mathrm{AUM} = \frac{\mathrm{Cov}[T, \Lambda(T)]}{\sigma_T \cdot \sigma_{\Lambda(T)}} = \frac{e^{\dagger}}{\sigma_T}\,,$$

which must therefore be strictly positive since $e^{\dagger} > 0$ and $\sigma_T > 0$ when T admits a probability density function. Hence $0 < \mathrm{AUM} \leq 1$, and therefore it follows that $0 < e^{\dagger} \leq \sigma_T$, which proves Equation (2).

Let us now derive the partial derivative of AUM with respect to calendar year y:

$$\dot{\mathrm{AUM}} = \frac{\dot{e}^{\dagger}\,\sigma_T - \dot{\sigma}_T\, e^{\dagger}}{\sigma_T^2}\,, \tag{6}$$

from which it follows that the partial (and relative) derivative of AUM is positive (negative) when the numerator of Equation (6) is positive (negative). Since $\log(\dot{\mathrm{AUM}}) = \dot{\mathrm{AUM}}/\mathrm{AUM}$, and assuming that both σ_T and $e^{\dagger}$ are strictly positive, we immediately have

$$\begin{aligned}
\log(\dot{\mathrm{AUM}}) \lesseqgtr 0 \quad &\iff \quad \dot{\mathrm{AUM}} \lesseqgtr 0 \quad \iff \quad \dot{e}^{\dagger}\,\sigma_T - \dot{\sigma}_T\, e^{\dagger} \lesseqgtr 0 \\
&\iff \quad \frac{\dot{e}^{\dagger}}{e^{\dagger}} \lesseqgtr \frac{\dot{\sigma}_T}{\sigma_T} \quad \iff \quad \log(\dot{e}^{\dagger}) \lesseqgtr \log(\dot{\sigma}_T),
\end{aligned}$$

which proves Equation (3).

Given the definition of $\mathcal{H}$ and the newly found range for AUM, Equation (4) follows immediately.

Lastly, consider the case in which all subjects have the same initial exact age a and face identical force of mortality $\lambda(t)$ in future years $t \geq 0$ beyond a. We can generalize our results and definitions to this more general setting, so that, for example,

$$0 < e^{\dagger}(a) \leq \sigma_T(a)\,,$$

where $e^{\dagger}(a)$ is the life lost to deaths after age a, and $\sigma_T(a)$ is the standard deviation of time to death after age a, both conditional on survival to age a. Note that this scenario can be rewritten in terms of conditional random variable and conditional density function of the original population, starting from age 0 rather than age a (see Supplementary Materials).

3. Related results

The closest link to the relationships that we present in this paper is provided by Schmertmann (2020) in the framework of revivorship models, where the author derives the equality between $e^{\dagger}$ and the covariance between T and its transformation through its own cumulative hazard function.

A recent interesting use of the cumulative hazard function is provided by Ullrich, Schmertmann, and Rau (2022): The authors introduce a new longevity measure based on the cumulative hazard. The proposed death expectancy H_1 indicator corresponds to the age at which the cumulative hazard is equal to one (or equivalently, the survival function is about 36.8%). The authors argue that the H_1 measure could be used as a dynamic threshold age for the oldest-old.

In demography, there exists a variety of indicators that are used to summarize the age-at-death distribution. Several indicators focus on the first moment, or location, of the distribution – that is, the so-called central longevity indicators (mean, median, and modal ages at death) (Cheung et al. 2005; Canudas-Romo 2010). Another class of indicators is used to study the second moment, or scale, of the distribution. Both absolute and relative measures of the variability of the distribution are used for this purpose. Examples of absolute indices are the variance and the $e^{\dagger}$ measures, while the life-table entropy, the coefficient of variation, and the Gini coefficient are examples of relative measures. Typically, relative measures of variability are computed by dividing an absolute variability measure by life expectancy. Up to our knowledge, the AUM index is among the very first mortality indicators that go beyond these two classes of indicators, by analyzing the ratio of two absolute measures of variation. One recent proposal in this direction is the ratio of expansion to compression measure, which considers the $e^\dagger$ components before and after the threshold age at death (Zhang and Li 2020).

The life-table entropy measure can be partly related to the AUM index, since both measures share the same numerator. The life-table entropy is a relative measure of variability of the age-at-death distribution compared to life expectancy at birth (Leser 1955; Keyfitz 1977; Demetrius 1974; Rezaei and Yari 2021). The difference between AUM and $\mathcal{H}$ is that while the former standardizes by the standard deviation, the latter standardizes by the life expectancy at birth e_0. The difference in the denominator of the two measures results into two different interpretations of the indices: The entropy measures the (relative) variability of the distribution, while the AUM measures how close the distribution is from having constant mortality. Both indices can nonetheless be used for the study of lifespan inequality, with the AUM providing an innovative perspective in terms of normalized lifespan inequality. Another recent attempt in this direction is made by Permanyer and Shi (2022), who introduce normalized lifespan inequality to explicitly consider that life expectancy has been increasing at a faster pace than maximal length of life. For any given year, the authors compute normalized lifespan inequality by dividing lifespan inequality indices by their maximum value under an hypothetical distribution with life expectancy equal to the observed one.

4. Applications

Here we illustrate the use of the AUM index with an application to observed period and cohort death rates (obtained by dividing deaths by exposures), as well as to period life-table death rates. All data were retrieved from the Human Mortality Database (2024, henceforth HMD). Routines for deriving these results were developed in `R` (R Core Team 2023) and are available in the Supplementary Materials as well as in an open-access repository.[5] The analytical formulas that we derived and employed for the implementation of our routines are available in the Supplementary Materials. Unless otherwise specified, all mortality measures are computed from age zero.

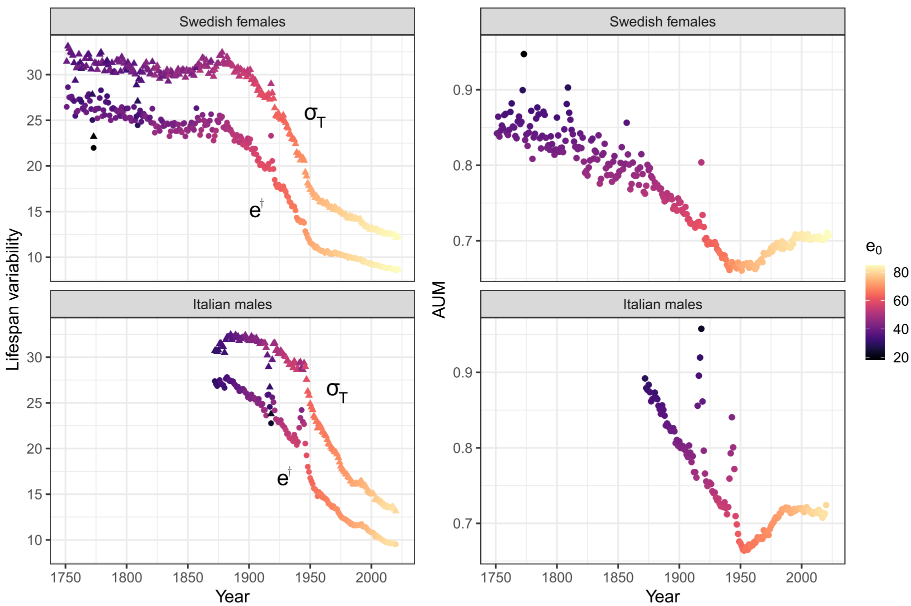
4.1 AUM at age 0

We start by investigating the temporal evolution of the AUM index in observed period death rates for two populations: Swedish females and Italian males. Figure 1 shows the $e^\dagger$, σ_T, and AUM measures over time for both populations. In the left panels, we can observe the well-known reduction of lifespan inequality over most of the period analyzed, coinciding with the increase in life expectancy (see, e.g., Edwards and Tuljapurkar 2005;

[5] Available at https://osf.io/fj94p/.

Wilmoth and Horiuchi 1999; Aburto et al. 2020). The two graphs show that $e^\dagger$ never exceeds σ_T, in agreement with our derived upper bound of $e^\dagger$. The right panels show that the AUM index declined rather consistently from 1751 until the 1950s, when it reached a minimum value. Thereafter, a reversal of the decreasing trend is observable. Sudden increases in the index are visible in correspondence with the Spanish flu (for both populations) and the two World Wars (for males only). From the 2000s, the index displays a rather constant behavior.

Figure 1: Evolution of the $e^\dagger$ (points) and σ_T (triangles) lifespan variability measures (left panels), and of the AUM index (right panels) over time for Swedish females and Italian males, 1751–2022. Colors correspond to different levels of life expectancy at birth (e_0)



Source (all figures): Authors' own elaborations on data from the HMD (2024).

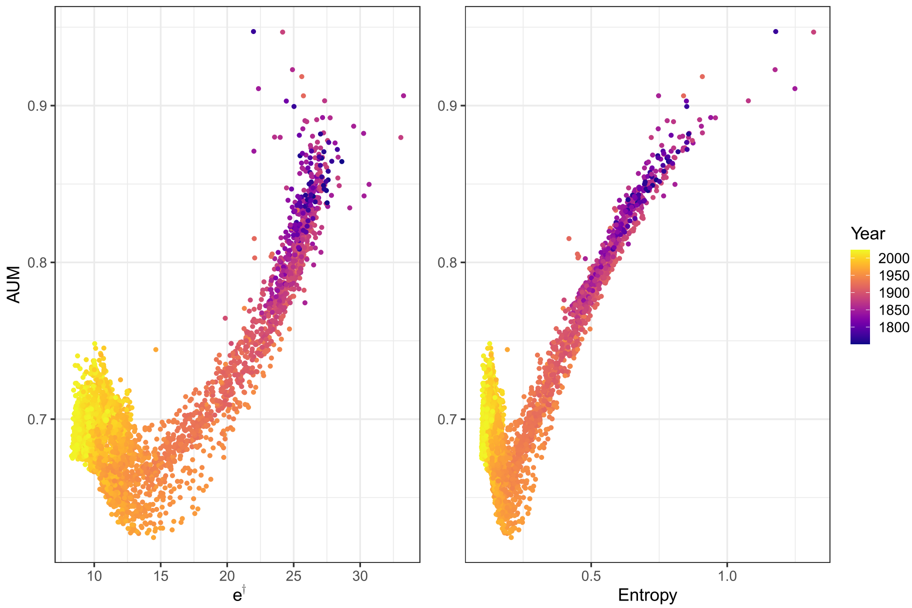
Analyzing the temporal trend of the AUM index across the 41 populations available in the HMD by sex provides additional insights. Figure 2 shows a rather substantial overlap in the decrease of the AUM index across sexes until the 1950s (except during the two World Wars). From the 1950s onward, as the AUM index stopped declining and started its increase, a marked departure from the overlap between sexes occurred, with male populations generally characterized by greater values of the AUM. It is further worth noticing that, for several female populations, the decline of the AUM halted and reversed somewhat later than the 1950s.

Figure 2: Evolution of the AUM index over time for 41 female (purple) and male (orange) populations, 1751–2023

How can we interpret these trends of the AUM index? Formally, as we have shown, the AUM decreases (increases) when the relative change in $e^{\dagger}$ is lower (greater) than the relative change in σ_T. This means that throughout most of the period analyzed (1751–2023), the relative change (generally, reduction) in σ_T was greater than the one in $e^{\dagger}$; however, a reversal of this trend occurred around the 1950s–60s. This period is often identified with a transition to a new mortality regime, characterized by an acceleration of mortality improvements at older ages (Kannisto et al. 1994; Vaupel et al. 1998; Wilmoth and Horiuchi 1999) and a more pronounced shifting dynamic of the age-at-death distribution (Bergeron-Boucher, Ebeling, and Canudas-Romo 2015). From this perspective, the AUM index provides insights on the transition from mortality compression to mortality shifting (Janssen and de Beer 2019).

An alternative interpretation of these findings can be made considering that the AUM is the normalized version of $e^\dagger$. The normalization implies that AUM takes values on a fixed support (i.e., between 0 and 1) for all years and populations. Conversely, $e^\dagger$ can vary on very different supports. Consider, for example, the top left panel of Figure 1. Until the 1900s, $e^\dagger$ could take values up to approximately 33 years; in 2000, its maximum value would have been approximately 13 years. The fact that $e^\dagger$ (and several other lifespan inequality measures) has a time-varying support may bias our assessment of lifespan inequality trends. Conversely, the fixed support of the AUM index allows for a more meaningful comparison of the index across countries and over time. Our findings suggest that the normalized years of life lost have not continued to decrease throughout the time period analysis (as suggested instead from the historical evolution of lifespan inequality measures, see, e.g., Edwards and Tuljapurkar 2005; Wilmoth and Horiuchi 1999). As lifespan inequality continued to decrease, normalized lifespan inequality started to increase around the 1950s–60s. This is further illustrated by Figure 3, which shows, for the 41 female populations of the HMD, the relationship between the AUM index and two indices of (absolute and relative) lifespan variability: the $e^\dagger$ and the entropy of the life table. The figure clearly shows that, as lifespan inequality continued to reduce over time, the AUM index declined for most of the period considered, reaching a minimum for values of $e^\dagger$ and the entropy around 13.5 and 0.2, respectively, when it then started to increase.

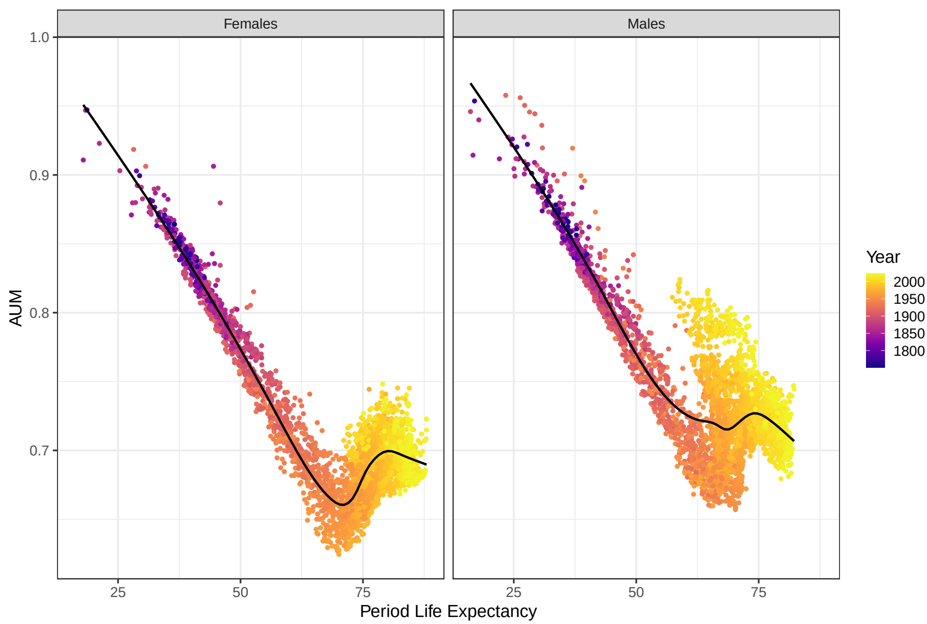
Figure 3: Relationship between the AUM index and absolute and relative lifespan variability, measured with the $e^{\dagger}$ (left) and the life-table entropy (right), respectively, for 41 female populations from 1751 to 2023. Colors correspond to different calendar years



The reduction of the AUM index throughout most of the analyzed period can also be interpreted with respect to how close the distribution is to an exponential one (i.e., one with constant mortality). On the one hand, the density of the exponential distribution is monotonically decreasing, having its maximum at age zero. On the other hand, the age-at-death distribution is typically bimodal, with one mode in infancy and another in late life (Kannisto 2001). It is probably not surprising that the highest values of the AUM index were observed in the past, when a significant number of deaths were occurring at infant and childhood ages: Indeed, the corresponding distribution of deaths was characterized by high and monotonically decreasing values at the youngest ages, somewhat closer to the shape (at least in those early ages) of the exponential distribution. Mortality improvements at these ages throughout subsequent decades decreased the relative importance of deaths at these ages, with more and more deaths occurring at older ages. These improvements in infant mortality are reflected in the reduction of the AUM.

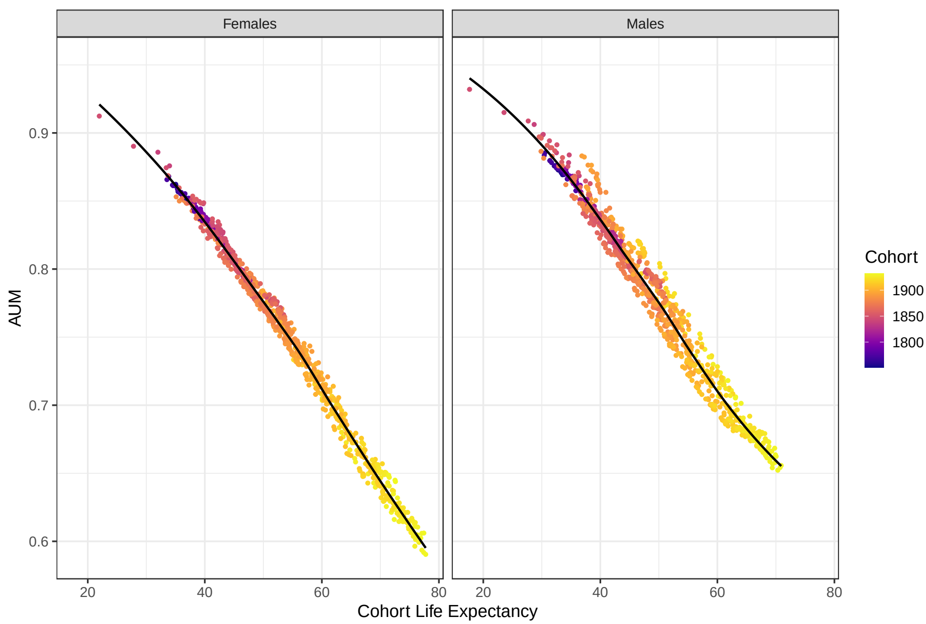
Next, we analyze the relationship between the AUM index and period life expectancy at birth. Figure 4 shows this relationship for the 41 populations in the HMD, by sex. For low levels of life expectancy, there is a linear negative relationship between the two measures. However, there appears to exist a threshold level of life expectancy (around age 70 and 60 for females and males, respectively) at which the negative relationship ceases to hold. For females, a positive relationship emerges above the threshold, whereas for males, the relationship appears to be more erratic. It should be noted that, due to data limitations regarding historical data, few data points are available for older periods, which are characterized by lower levels of life expectancy; as such, the strong linear relationship observed for low levels of life expectancy could be partially due to data limitations.

Figure 4: Relationship between the AUM index and period life expectancy at birth for 41 female (left) and male (right) populations from 1751 to 2023. Colors correspond to different calendar years. The black line corresponds to smoothing the observed data using a cubic regression spline (using the `gam` function of the `mgcv` package (Wood 2017))



We now move to the same analysis using cohort instead of period observed death rates. Figure 5 shows the relationship between the AUM index and cohort life expectancy at birth by sex for all ten HMD populations with cohort life-table data.[6] Unlike the period analysis, one observes a lack of reversal of the linear relationship between the two measures. For cohorts, the AUM tends to linearly decrease over increasing values of cohort life expectancy (typically belonging to the more recent cohorts). It is worth noting that this linear relationship is linked to the cohort nature of the data analyzed and not a result of country selection: Indeed, the period analysis of this relationship for the ten countries shows a very similar pattern to that reported in Figure 4, with a break in the linear relationship at higher levels of life expectancy (see Figure S1 in the Supplementary Materials).

Figure 5: Relationship between the AUM index and cohort life expectancy at birth for ten female (left) and male (right) populations from 1751 to 1932. Colors correspond to different birth cohorts. The black line corresponds to smoothing the observed data using a cubic regression spline (using the `gam` function of the `mgcv` package (Wood 2017))



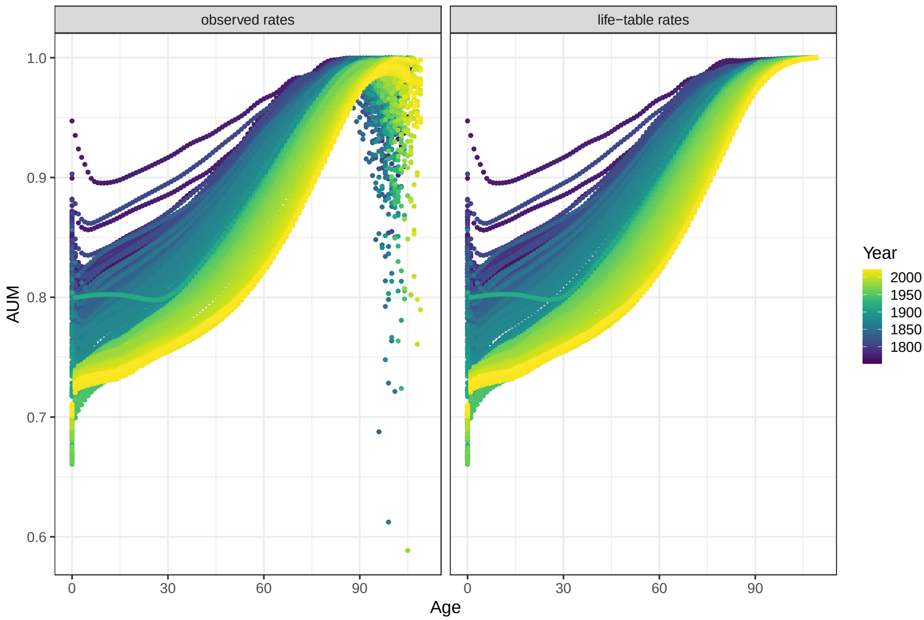
[6] The ten populations are those of France, Italy, Finland, Denmark, Sweden, the Netherlands, Iceland, Norway, Switzerland, and England and Wales.

4.2 AUM at all ages

Finally, we turn to study the behavior of the AUM index at all ages, focusing on a single population only, namely, Swedish females. In addition to computing the AUM using observed death rates, we also employ period life-table death rates. The reason for doing so is that observed and life-table death rates differ at the oldest ages, since the latter are smoothed at older ages using the Kannisto model of mortality (see Wilmoth et al. 2021: p. 34).

Figure 6 shows the results of the age analysis. In general, we observe that the AUM index tends to increase with age (declining in the first few years of life in the oldest periods considered). Importantly, the figure highlights the difference of the AUM index at the oldest ages. When the index is computed on observed rates (left panel), there is high variability of the AUM estimate at the oldest ages, reflecting the variability of the underlying rates; conversely, when life-table death rates are used as input, the AUM index approaches the value of 1. This was an expected result since life-table death rates are smoothed at older ages according to a logistic pattern (the Kannisto model), which is characterized by a hazard function that becomes increasingly flatter, thus mimicking an exponential (constant) hazard (whose AUM value is exactly 1).

Figure 6: AUM index over ages 0 to 110+ for Swedish females for the years 1751–2022 computed on observed rates (left panel) and life-table rates (right panel)



It is worth mentioning here that conventional routines generally employed to calculate life-table variability measures (such as those available in the `LifeIneq` R package, Riffe, van Raalte, and Dudel 2023) return AUM estimates that exceed its upper bound at the very oldest ages (i.e., AUM values greater than 1), suggesting some potential bias in the estimation of $e^\dagger$ or σ_T, or both, at older ages. Please refer to the Supplementary Materials for a comparison analysis between our AUM estimates and those derived from conventional routines.

5. Discussion

In this paper, we have elaborated on the $e^\dagger$ measure of lifespan inequality introduced by Vaupel and Canudas-Romo (2003). Leveraging a recent result noted in Schmertmann (2020), we derived the upper bound of $e^\dagger$. This is, to our knowledge, a novel and important result. It is intriguing that the upper bound of $e^\dagger$ is another absolute measure of lifespan inequality – the standard deviation of ages at death in the population (σ_T). Even more intriguing, we have shown that $e^\dagger$ reaches its upper bound if and only if the underlying age-at-death distribution is exponential.

The upper bound of $e^\dagger$ stimulated us to introduce the 'Average Uneven Mortality' (AUM) index, a new mortality index that can be used to study mortality across age and over time. The index has two closely related interpretations. On the one hand, it measures the linearity of the relationship between the random variable T and its cumulative hazard function; on the other hand, and equivalently, it measures the distance of the age-at-death distribution from an exponential one, or the distance of the hazard function from a constant ('even') hazard. The AUM is a relative index, bounded between 0 and 1, and it is obtained by dividing two absolute measures of variation of the age-at-death distribution. This is one among the very first proposals to build a mortality indicator based on (the ratio of) two lifespan inequality measures. Other relative indicators of mortality employed in the literature are computed by dividing an absolute variability measure by life expectancy (e.g., life-table entropy, coefficient of variation, Gini coefficient).

Introducing a novel mortality index raises the question 'So what?'. We believe that the AUM index provides novel insights on the study of human mortality age patterns and time developments. Importantly, the AUM index takes values on a fixed support, suggesting that comparison across countries and over time are more meaningful than when one uses an indicator whose range can change over time (such as $e^\dagger$). Indeed, in applied statistics, the correlation coefficient is generally favoured to the covariance, since the latter tells little about the strength of the dependence between two random variables. The analysis of the normalized $e^\dagger$ suggests that the decrease of lifespan inequality has reversed its secular decline in most recent decades. Our findings align well with those of Permanyer and Shi (2022), who also observed that declines of normalized lifespan inequality stopped and even reversed at high levels of life expectancy. Furthermore, while it is well known that lifespan inequality measures are highly correlated between each other (see, e.g., Wilmoth and Horiuchi 1999; Van Raalte and Caswell 2013), the evolution of the AUM index shows that the relative change of $e^\dagger$ and σ_T differed over time: The relative change (typically reduction) in σ_T was greater than the one of $e^\dagger$ for most of the period that we analyzed. The 1950s marked a clear reversal of this trend, likely connected to accelerating mortality improvements at older ages and related shifting of the age-at-death distribution.

The relationship between the AUM index and life expectancy at birth provides an interesting perspective on the evolution of the age-at-death distribution. We found a linear negative relationship between the AUM and life expectancy, which ceased to hold only at high levels of period life expectancy. Interestingly, this disruption did not occur in the analysis of cohort mortality data. Due to the limitation of such data, we cannot know whether this disruption will materialize for more recent birth cohorts, or if it is a specific feature of the most recent period death rates. If the second hypothesis were to hold true, it would imply that information on the scale of the age-at-death distribution would be predictive of its location, with important consequences for mortality analysis and forecasting.

The AUM index can also be employed to study the mortality age pattern. In our analysis, we observed that the AUM index generally increases over age. In the analysis of period life tables (with modeled death rates), the index always approaches its upper limit at older ages, signaling that the hazard function resembles a constant exponential hazard. When observed death rates are employed instead, there is significant variability in the estimate of the AUM at older ages. Clearly, employing the AUM index for detecting mortality deceleration, and eventually the existence of a mortality plateau, has great potential; our proposed indicator could thus contribute to the current debate about the mortality plateau at the oldest ages (see, e.g., Barbi et al. 2018; Dang et al. 2023; Gampe 2021; Newman 2018). The quantification of the statistical uncertainty associated with the estimated AUM estimate should be a critical aspect to inform such analysis, especially for small sample sizes, and we plan to pursue this in our future research.

An important contribution of our work relates to the software routines that we have developed to calculate the AUM index. Initially, we started by computing the AUM using standard and available routines for calculating lifespan inequality measures from a life table. The results that we obtained were unexpected as the AUM was exceeding its upper bounds at older ages – an empirical result that contradicted our theoretical findings. As such, we decided to implement new routines based on the formulas that we derived in this paper. The new empirical estimates that we obtained did not present such anomalies. Evidence presented in the Supplementary Materials suggests that our routines for computing $e^\dagger$ and σ_T improve estimation precision with respect to conventional routines, particularly for the older ages. Our routines are publicly available in the Supplementary Materials accompanying this article as well as in a public repository, and we hope that further computational efforts will be directed to assess estimation accuracy of lifespan variability measures at the oldest ages.

Given the definitions of AUM and of the entropy index $\mathcal{H}$, if two populations have equal $e^\dagger$ but different AUM, that information would be equivalent to an examination of the difference in their standard deviations. This is similar to the conclusion that one would draw if two populations have same $e^\dagger$ but different $\mathcal{H}$, with respect to differences in life expectancy between the two populations. Combining these two observations yields the fact that if two populations have different values for AUM but equal $\mathcal{H}$ (or the other way around), that should therefore be attributed to a difference in the coefficients of variation of the two populations. In general, it seems clear that an understanding of the survival pattern of a population will benefit from the examination of all these indices.

Lastly, note that measuring $e^\dagger$ in standard deviation units through AUM improves the interpretation of inequality in mortality by allowing one to highlight novel shape behaviors. Indeed, the AUM index can be related to the more general study of the shape of the age-at-death distribution, which has gained increasing attention in most recent decades (for a recent review, see, e.g., Bonetti, Gigliarano, and Basellini 2021). In particular, as we showed, the AUM index approaches the value of one if and only if the tail behavior is that of an exponential random variable, regardless of the value of the parameter θ of the distribution. Such invariance with respect to θ ensures that the phenomenon of increasing AUM to its upper bound is not equivalent to the simple decrease in the standard deviation of the death ages (which is equal to $1/\theta$ for the exponential model). We believe that this novel mortality indicator can provide additional insights on human mortality, enlarging the toolbox of available methods for the analysis of mortality developments.

6. Acknowledgements

The authors would like to thank José Manuel Aburto, the Guest Editor Carl Schmertmann, and two anonymous reviewers for their comments on a previous version of this manuscript. UB would like to thank Jim Vaupel for his kindness and generosity, for his support and encouragement to become a demographer, and for his inestimable contributions to our discipline – we all miss you very much.

7. Author statement

The first two authors contributed equally to the paper. Conceptualization: MB, UB, AN. Methodology: MB. Software: MB, UB. Data curation and visualization: UB. Supervision: MB. Writing original draft: MB, UB. Reviewing and editing: MB, UB, AN.

8. Funding

Marco Bonetti and Andrea Nigri were supported by the MUR-PRIN 2022 project CARONTE (Prot. 2022KBTEBN), funded by the European Union – Next Generation EU. MB was also supported by the MUSA – Multilayered Urban Sustainability Action – project, funded by the European Union – Next Generation EU, under the National Recovery and Resilience Plan (NRRP) Mission 4 Component 2 Investment Line 1.5.

References

Aburto, J.M., Villavicencio, F., Basellini, U., Kjærgaard, S., and Vaupel, J.W. (2020). Dynamics of life expectancy and life span equality. *Proceedings of the National Academy of Sciences* 117(10): 5250–5259. doi:10.1073/pnas.1915884117.

Barbi, E., Lagona, F., Marsili, M., Vaupel, J.W., and Wachter, K.W. (2018). The plateau of human mortality: Demography of longevity pioneers. *Science* 360(6396): 1459–1461. doi:10.1126/science.aat3119.

Bergeron-Boucher, M.P., Ebeling, M., and Canudas-Romo, V. (2015). Decomposing changes in life expectancy: Compression versus shifting mortality. *Demographic Research* 33(14): 391–424. doi:10.4054/DemRes.2015.33.14.

Bonetti, M., Gigliarano, C., and Basellini, U. (2021). The Gini Concentration Index for the study of survival. In: Mukhopadhyay, N. and Sengupta, P.P. (eds.). *Gini Inequality Index*. London: Chapman and Hall/CRC: 107–124. doi:10.1201/9781003143642-6.

Canudas-Romo, V. (2010). Three measures of longevity: Time trends and record values. *Demography* 47(2): 299–312. doi:10.1353/dem.0.0098.

Cheung, S.L.K., Robine, J.M., Tu, E.J.C., and Caselli, G. (2005). Three dimensions of the survival curve: Horizontalization, verticalization, and longevity extension. *Demography* 42(2): 243–258. doi:10.1353/dem.2005.0012.

Dang, L.H.K., Camarda, C.G., Meslé, F., Ouellette, N., Robine, J.M., and Vallin, J. (2023). The question of the human mortality plateau: Contrasting insights by longevity pioneers. *Demographic Research* 48(11): 321–338. doi:10.4054/DemRes.2023.48.11.

Demetrius, L. (1974). Demographic parameters and natural selection. *Proceedings of the National Academy of Sciences* 71(12): 4645–4647. doi:10.1073/pnas.71.12.4645.

Edwards, R.D. and Tuljapurkar, S. (2005). Inequality in life spans and a new perspective on mortality convergence across industrialized countries. *Population and Development Review* 31(4): 645–674. doi:10.1111/j.1728-4457.2005.00092.x.

Gampe, J. (2021). Mortality of supercentenarians: Estimates from the updated IDL. In: Maier, H., Jeune, B., and Vaupel, J.W. (eds.). *Exceptional lifespans*. Cham: Springer International Publishing: 29–35. doi:10.1007/978-3-030-49970-9_3.

Human Mortality Database (2024). University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de. Data retrieved on March 27, 2024.

Janssen, F. and de Beer, J. (2019). The timing of the transition from mortality compression to mortality delay in Europe, Japan and the United States. *Genus* 75: 1–23. doi:10.1186/s41118-019-0057-y.

Kannisto, V. (2001). Mode and dispersion of the length of life. *Population: An English Selection* 159–171. https://www.jstor.org/stable/3030264.

Kannisto, V., Lauritsen, J., Thatcher, A.R., and Vaupel, J.W. (1994). Reductions in mortality at advanced ages: Several decades of evidence from 27 countries. *Population and Development Review* 793–810. doi:10.2307/2137662.

Keyfitz, N. (1977). What difference would it make if cancer were eradicated? An examination of the Taeuber paradox. *Demography* 14: 411–418. doi:10.2307/2060587.

Leser, C.E.V. (1955). Variations in mortality and life expectation. *Population Studies* 9(1): 67–71. doi:10.2307/2172342.

Newman, S.J. (2018). Errors as a primary cause of late-life mortality deceleration and plateaus. *PLoS Biology* 16(12): e2006776. doi:10.1371/journal.pbio.2006776.

Permanyer, I. and Shi, J. (2022). Normalized lifespan inequality: Disentangling the longevity–lifespan variability nexus. *Genus* 78(1): 1–29. doi:10.1186/s41118-021-00150-6.

R Core Team (2023). R: A language and environment for statistical computing. Vienna: R Foundation for Statistical Computing. https://www.R-project.org.

Rezaei, R. and Yari, G. (2021). Keyfitz entropy: Investigating some mathematical properties and its application for estimating survival function in life table. *Mathematical Sciences* 15: 229–240. doi:10.1007/s40096-020-00354-5.

Riffe, T., van Raalte, A., and Dudel, C. (2023). LifeIneq: An R-package to calculate measures of lifespan variation from a life table. Available at https://github.com/alysonvanraalte/LifeIneq.

Schmertmann, C.P. (2020). Revivorship and life lost to mortality. *Demographic Research* S8(17): 497–512. doi:10.4054/DemRes.2020.42.17.

Ullrich, N., Schmertmann, C., and Rau, R. (2022). Death expectancy H_1: An alternative measure for longevity. Paper presented at PAA 2022 Annual Meeting, Atlanta, Georgia and online, April 6–9, 2022.

Van Raalte, A.A. and Caswell, H. (2013). Perturbation analysis of indices of lifespan variability. *Demography* 50(5): 1615–1640. doi:10.1007/s13524-013-0223-3.

Vaupel, J.W. and Canudas-Romo, V. (2003). Decomposing change in life expectancy: A bouquet of formulas in honor of Nathan Keyfitz's 90th birthday. *Demography* 40(2): 201–216. doi:10.1353/dem.2003.0018.

Vaupel, J.W., Carey, J.R., Christensen, K., Johnson, T.E., Yashin, A.I., Holm, N.V., Iachine, I.A., Kannisto, V., Khazaeli, A.A., Liedo, P., Longo, V.D., Zeng, Y., Manton, K.G., and Curtsinger, J.W. (1998). Biodemographic trajectories of longevity. *Science* 280(5365): 855–860. doi:10.1126/science.280.5365.855.

Wilmoth, J.R., Andreev, K., Jdanov, D., Glei, D.A., Riffe, T., Boe, C., Bubenheim, M., Philipov, D., Shkolnikov, V., Vachon, P., Winat, C., and Barbieri, M. (2021) Methods protocol for the Human Mortality Database. Last Revised: January 26, 2021 (Version 6).

Wilmoth, J.R. and Horiuchi, S. (1999). Rectangularization revisited: Variability of age at death within human populations. *Demography* 36(4): 475–495. doi:10.2307/2648085.

Wood, S. (2017). *Generalized additive models: An introduction with R*. London: Chapman and Hall/CRC, 2 ed.

Zhang, Z. and Li, Q. (2020). The ratio of expansion to compression: A new measure of lifespan disparity. *PLoS One* 15(12): e0243482. doi:10.1371/journal.pone.0243482.