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Research Article

Fertility quantum and tempo with cubic age-specific birth rates

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Robert Schoen[1]

Abstract

OBJECTIVE
To demonstrate the analytical value of a cubic parameterization of the age curve of fertility and to explore its features, especially its usefulness in separating fertility level and fertility timing.

METHODS
Using mathematical analysis, the cubic fertility curve is derived and examined in both continuous and discrete forms.

RESULTS
The cubic curve for replacement level fertility is found and expressed in terms of the mean age of fertility. That baseline cubic birth rate density, proportionately adjusted for the level of fertility, is shown to plausibly fit observed birth rates and imply a new approximation for their implicit stable growth rate. Because the proposed cubic model separates the effects of fertility level (quantum) and fertility timing (tempo), it leads to new period/cohort and population momentum relationships and provides a structure for relating fertility trajectories to birth sequences in changing rate models.

CONTRIBUTION
The cubic parameterization can simplify the representation of age curves of fertility rates while capturing their essential features. With a proportional adjustment at all ages to reflect fertility level, the cubic model can separate level and timing effects and permit numerous analytical applications. Of note, those applications include a new and superior approach to how changes in period tempo with constant quantum affect cohort fertility.

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1. Introduction

Population dynamics reflect the complex interaction between the level and pattern of age-specific fertility and population size and composition. Here we derive relationships based on age-specific fertility that follows a cubic curve and analyze their implications for the level and timing of fertility.

Age-specific birth rate (ASBR) curves show great regularities. Fitting, or parameterizing, the curve to a known function not only serves to summarize it but also facilitates analyses of fertility. The three classic efforts used the normal curve (Lotka 1939), the Pearson Type III (gamma) curve (Wicksell 1931), and a convolution of exponentials (Hadwiger 1940). All three were based on moments of the population's ASBRs: (a) the zeroth moment, or the level of fertility; (b) the first moment, or mean age of fertility; and (c) the second moment about the mean, or the variance. Keyfitz (1977, Chapter 6) analyzed and compared the three, finding that all were able to yield an accurate value for Lotka's intrinsic growth rate (r) but were less successful in tracking empirical ASBR curves. Coale and Trussell (1974) developed three-parameter model fertility schedules in seeking to capture the full range of age-specific fertility in large populations. Coale and McNeil (1972) derived a three-parameter model of first marriage, which Bloom (1982) used to examine patterns of first births and to estimate childlessness. Hoem et al. (1981) examined the fit of eight different parametric functions, including the gamma, Hadwiger, and Coale–Trussell, which were judged to fit reasonably well. Duchene and Gillet-de Stefano (1974) and Pressat (1995) considered simpler cubic curves, though they found satisfactory fits only for very high-fertility populations.

Here, the focus is not on goodness of fit but on mathematical tractability for analytical experimentation. Schoen (2022) used a simple quadratic to approximate the ASBR curve in an analysis relating period and cohort fertility. While useful for that purpose, the quadratic cannot reflect the skewness of the typical ASBR curve. Cubic curves allow greater flexibility, are easy to integrate and differentiate, and can plausibly approximate observed age-specific birth rate curves. A cubic curve is derived here, and its properties and suitability are examined. Most importantly, the flexibility of the cubic curve allows new opportunities for analyzing relationships between fertility level (or quantum) and fertility timing (or tempo), including the ability to change period tempo without changing period quantum and to gauge timing effects on population momentum.

Contributions of the paper include (1) specifying a simple yet flexible way to describe the age curve of human fertility, (2) separating the effects of timing and level change on overall fertility, (3) providing a new relationship between fertility and implicit stable population growth, and (4) offering a new analytical tool for fertility analysis, whose applications include period/cohort translations and the impact of tempo on population momentum.

2. Specifying the cubic curve of age-specific birth rates

To start, we specify a fixed cubic density function to describe the age pattern of replacement level fertility. We then consider changes in the level of fertility that preserve those proportions at each age. We examine one-sex (female) models, with no mortality below age 45, the highest age of reproduction. Mortality under age 45 is quite low in many contemporary populations. For example, in the United States Life Tables for Females in 2015, 96.9% of births survive to age 45 (Arias and Xu 2018: 13). Ignoring mortality simplifies the calculations and makes the results more transparent.

Let f(x) be the base age-specific occurrence/exposure net maternity (here also fertility) rate at age x, with

$$1 = \int f(x)\, dx, \tag{1}$$

indicating a base fertility level (density), whose net (or gross) reproduction rate (R) is 1. Because all births are assumed to survive to age 45, the integral ranges over all ages of reproduction, here taken to be ages 15 to 45. Unless otherwise noted, that age range applies to all integrals considered.

We assume that fertility changes proportionally at all ages, so when the net reproduction rate (NRR) is R, age-specific fertility is given by R f(x), and we have the identity

$$R = \int R\, f(x)\, dx \tag{2}$$

Proportional change is both simple and empirically common (e.g., Hobcraft, Menken, and Preston 1982) and allows the quantum and tempo of fertility to move independently of one another. Do note that here f(x) is the fixed fraction of fertility (or net reproduction or net maternity) at exact age x, with the ASBR at age x equal to [R f(x)].

2.1 The cubic fertility density curve

Let the base age curve of fertility be given by the cubic equation

$$f(x) = a + bx + cx^2 + dx^3 \tag{3}$$

To determine the four cubic coefficients (or parameters), one must solve a set of four equations. Three of them are structurally specified, as the curve must equal 0 at age 15 and age 45, and the area under the curve between those ages must equal 1. Hence there is only *one* choice parameter, which can be taken to be mean age μ. The specifying equations are then

$$f(15) = 0 = a + 15b + 225c + 3375d$$
$$f(45) = 0 = a + 45b + 2025c + 91{,}125d$$
$$1 = \int [a + bx + cx^2 + dx^3]\, dx$$
$$\mu = \int x\, [a + bx + cx^2 + dx^3]\, dx \qquad (4)$$

Solving those four simultaneous equations in terms of μ, as described in Appendix A, yields the desired cubic density function:

$$f(x) = (-63/20+\mu/10) + x(19/50 - 11\mu/900) + $$
$$+x^2(-61/4500 + \mu/2250) + x^3(1/6750 - \mu/202500) \qquad (5)$$

When μ = 30, the cubic curve reduces to the quadratic curve used in Schoen (2022):

$$f(x) = -3/20 + x/75 - x^2/4500 \qquad (6)$$

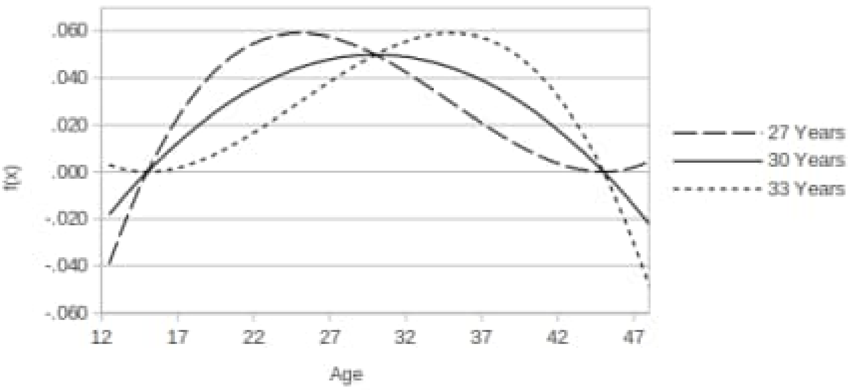
Equation (5) provides the desired base cubic curve as a function of μ. Using Equation (2), it can provide the ASBRs for any level of fertility. Thus the cubic ASBR curve depends on the zeroth and first moments alone.

2.2 Features of the continuous cubic density function

For f(x) to be a proper density function, all its values must be non-negative over the 15 to 45 age range. As shown in Figure 1, that holds true for μ between 27 and 33 years, the typical mean ages of fertility. Over that interval, the curve is unimodal and the skewness is apparent, especially for μ equaling 27 and 33. The peak of the curve is derived in Appendix B. As shown in Table 1, the peak age varies from age 25 when μ = 27 to age 35 when μ = 33. Outside that 27 to 33 range of μ, however, some values of f(x) become negative.

When that age range is inappropriate, for example because of high fertility at later ages, the 30-year reproductive age interval can be shifted upward, say to span ages 17.5 to 47.5. Acceptable μ values would then be in the 29.5 to 35.5 interval. The derivation would follow the same lines as in Appendix A, but the parameter values in Equations (3)–(6) would change.

Figure 1: Plots of cubic density function f(x) for mean age of fertility equal to 27, 30, and 33 years



Using the standard statistical definition, the variance of the cubic f(x) is given by

$$\text{Var} = \int x^2 f(x)\, dx - \mu^2 \qquad (7)$$

Carrying out the integration with f(x) given by Equation (5), we have

$$\text{Var} = 60\mu - 855 - \mu^2 \qquad (8)$$

In short, the variance of f(x) is a simple quadratic function of the mean age of fertility.

The skewness of f(x) is given by the third moment about the mean. From the derivation in Appendix C, it can be written

$$\text{Skew} = -52842.865 + 5361.4286\mu - 180\mu^2 + 2\mu^3 \qquad (9)$$

Table 1 shows values of the cubic density function for a range of mean ages of fertility. The variance is at a maximum, and the skewness at a minimum, when $\mu = 30$. Coefficients a and c increase with μ, while coefficients b and d decrease. The change in a is linear, while the values of d above and below $\mu = 30$ are equal but of opposite sign.

Table 1: Values of cubic density function f(x) for mean ages of fertility (μ) equal to 27, 29, 30, 31, and 33

Mean age (μ)	Peak age	Variance	Skewness	Coefficients of the cubic function *a*	*b*	*c*	*d*
27	25	36	61.7	–.45	.05	−.001556	.0000148
29	27.68	44	36.6	–.25	.025556	−.000667	.00000494
30	30	45	0	–.15	.013333	−.000222	0
31	32.32	44	–36.6	–.05	.001111	.000222	−.00000494
33	35	36	–61.7	.15	−.023333	.001111	−.0000148

Note: Values calculated from Equations (3), (7), (8), (9), and (A.2).

2.3 Relationships to the implicit stable population model

Under stable population theory (Lotka 1939; Keyfitz 1977), if any set of ASBRs prevails indefinitely, a stable population will emerge, one with a fixed age composition and constant growth at intrinsic rate r. While the cubic model expounded here is not stable because R can vary over time, at every time point the prevailing rates do imply a long-term stable population. To find the intrinsic r from the moments of a given fertility distribution, Lotka derived the approximate but quite accurate relationship

$$r = [\ \mu - \mathrm{SQRT}(\mu^2 - 2\ \mathrm{Var}\ \ln R)]\ /\ \mathrm{Var}, \tag{10}$$

where SQRT indicates the square root and ln the natural logarithm (see Keyfitz 1977: 142).

Keyfitz (1977: 118, 126) also showed that the same relationship, to second moments, follows from the renewal equation and a series expression for T, Lotka's generation length, which is defined by R = exp[rT].

Since Equation (8) and the fixed f(x) assumption give the variance as a function of μ, we can write a new approximate relationship for intrinsic r:

$$r = [\ \mu - \mathrm{SQRT}(\mu^2 - 2\ \ln R\ \{60\mu - 855 - \mu^2\ \})]\ /\ \{60\mu - 855 - \mu^2\}, \tag{11}$$

which expresses Lotka's r in terms of R and μ alone.

Although r is approximately ln(R)/μ, that confounds μ and T, two conceptually and theoretically different quantities. A better approximation than ln(R)/μ, which gives values quite close to Equation (11), was presented by (McCann 1973), but McCann's relationship is purely empirical and lacks any theoretical basis. Fundamentally, there are three mean ages relevant to fertility measurement: mean age of fertility μ; generation length T; and the mean age of childbearing A*, whose weights are births rather than birth rates. Generation length T is approximately the mean of μ and A* (Keyfitz 1977: 126). While μ and T are often similar in value, they may differ substantially under plausible demographic conditions. For example, if r = 0.04 and Var = 55, μ and T may differ by more than a full year.

Using Equations (8) and (11), we can examine how r changes with R. Differentiating Equation (11) with respect to R yields

$$\partial r/\partial R = 1 / [R \text{ SQRT}(\mu^2 - 2 \ln(R) \text{ Var})] \quad (12)$$

Rearranging and again using Equations (8) and (11):

$$\partial r/\partial R = 1 / [R(\mu - r \text{ Var})] \quad (13)$$

Equation (13) is a new relationship that quantifies, for demographically reasonable parameters, how r increases more when r and Var are larger and when R and μ are smaller.

Table 2 shows how intrinsic r changes as R goes from 0.5 to 2.0 and μ goes from 27 to 33 years. With below replacement fertility, r increases (becomes less negative) by 22% as μ increases from 27 to 33 years. At replacement level, timing does not matter as there is no growth. When R = 2, intrinsic growth falls by 19% when μ goes from 27 to 33 years.

Table 2: Variation of stable growth rate (r) and its associated doubling [halving] time with fertility level R and cubic fertility mean μ

Mean	Fertility level (R)			
(μ)	0.5	1.0	1.5	2.0
27	−.02525 [27.45]	0	.01517 [45.68]	.02613 [26.52]
30	−.02272 [30.50]	0	.01366 [50.75]	.02352 [29.46]
33	−.02077 [33.37]	0	.01237 [56.02]	.02125 [32.61]

Note: In each cell of the table, the first entry provides the stable rate r and the second (bracketed) entry the number of years for the population to double (or fall by half). That time, t_2, is calculated from $t_2 = \ln 2 / r$.

The second entry in each cell of Table 2 shows the number of years for the population to double in size (or fall by half). Doubling time increases from six to more than ten years as μ rises from 27 to 33 years. While the effects of level are clearly greater, fertility timing has a substantial influence on long-term population growth.

2.4 The discrete cubic formulation

The continuous cubic relationship in Equation (5) can be recast in discrete form. Here we do so in the context of five-year age groups, from 0–4 through 40–44. The process is usefully separated into two distinct steps.

The first step integrates the cubic density function over five-year intervals, from 15–19 through 40–44. With R = 1 and denoting the five-year fertility rates by φ(x,5), we can particularize Equation (2) and write

$$\varphi(x,5) = \int f(a)da, \qquad (14)$$

where the integral ranges from age x to age x + 5. The second step transforms the φ(x,5) rates to Leslie matrix elements F(x,5), following the approach in Preston, Heuveline, and Guillot (2001: 130). Here the subdiagonal elements of the Leslie matrix are all equal to 1, and the first two elements of the first row, representing ages 0–4 and 5–9, are 0. The third through ninth first-row elements, F(10,5) through F(40,5), are found from

$$F(x,5) = [\varphi(x, 5) + \varphi(x+5, 5)] / 2, \qquad (15)$$

where φ(10,5) and φ(45,5) are 0.

Table 3 gives the φ(x,5) and F(x,5) values as functions of μ and also shows F(x,5) values for mean ages of fertility 27, 30, and 33 (see Appendix D). The fertility values in the φ and μ columns allow investigators to calculate φ and F for any mean age from 27 to 33. Under the cubic parameterization, F(25,5) is always equal to 0.24074. Fertility values over age 30 increase with larger values of μ, while fertility values under 25 decrease. For mean values a given distance above and below age 30, the Leslie matrix values are identical, but with the ages reversed.

Table 3: Discrete fertility replacement level values and Leslie (projection) matrix first-row elements for five-year age intervals, in terms of mean age of fertility μ

Age interval	Replacement level values [φ(x,5)]	Leslie matrix values [F(x,5)] for mean ages 27≤ μ ≤ 33	27	30	33
10–14	0	.32639−.009645μ	.06597	.03704	.00810
15–19	.65278−.01929μ	.87037−.02469μ	.20370	.12963	.05556
20–24	1.08796−.03009μ	.86111−.02160μ	.27778	.21296	.14815
25–29	.63426−.01312μ	.24074	.24074	.24074	.24074
30–34	−.15278+.01312μ	−.43519+.02160μ	.14815	.21296	.27778
35–39	−.71759+.03009μ	−.61111+02469μ	.05556	.12963	.20370
40–44	−.50463+.01929μ	−.25231+.009645μ	.00810	.03704	.06597
Total	1	1	1	1	1

Note: Calculated as described in Appendix D.

3. Comparing cubic curve values to observed data

We need to examine if the cubic fertility curve adequately captures the nature of observed fertility rates. The emphasis is not on goodness of fit but on whether a cubic reflects the patterns of observed ASBR curves sufficiently well to support analytical explorations of fertility quantum and tempo. To illustrate that it does for a few contemporary low- to medium-fertility populations, Figure 2 plots published five-year net maternity values against cubic φ(x,5) values for the United States in 1985 (from Keyfitz and Flieger 1990), and Figure 3 shows five-year birth rates for the United States in 2020 (from Osterman et al. 2022) against comparable cubic values.

Figure 2: Plots of observed and cubic five-year birth rates [φ (x,5)] for the United States, 1985

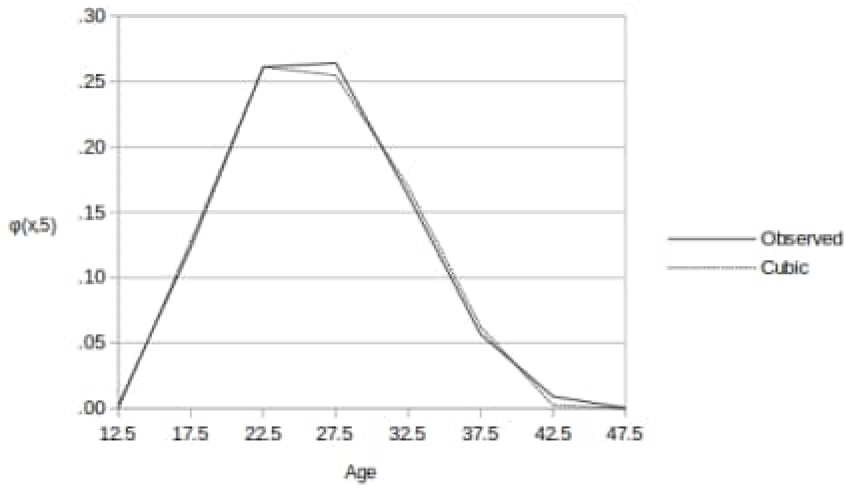
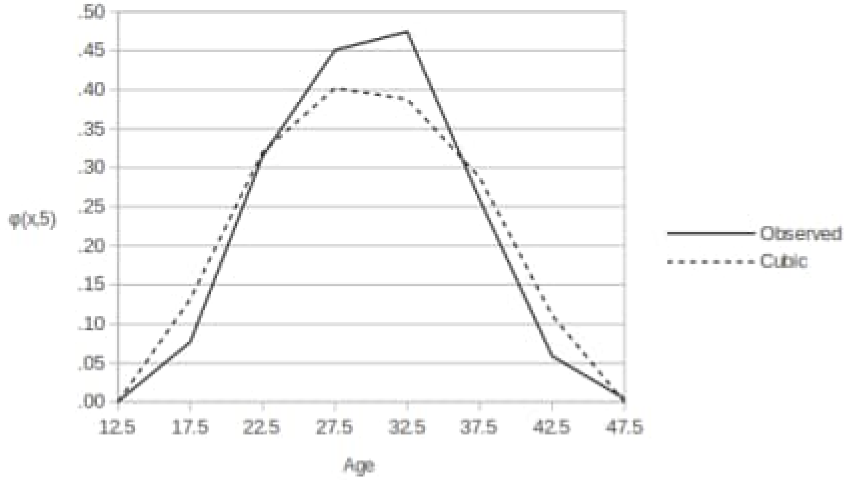


Figure 2 indicates rather close agreement between the cubic and observed curves for the United States in 1985, while Figure 3 reveals rather poor agreement for the United States in 2020. Those results are consistent with calculations for other populations (not shown) and indicate the strengths and weaknesses of the cubic parameterization. The cubic can capture the overall level and mean of the fertility curves and approximate their skewness. For broad peak patterns, as in the United States in 1985, where fertility at ages 20–24 and 25–29 is roughly equal, agreement between observed and cubic is quite good. For single peak patterns, like those in the United States in 2020, the fit is not very good, as the cubic is too low at the peak fertility age and too heavy in the tails of the distributions. With only two parameters, the cubic cannot capture observed changes in the variance of fertility, making it vulnerable to such departures as it overestimates the variance. However, for many populations and many purposes, including those of interest here, the cubic curve offers a satisfactory approximation because the effects of the variance are of minor consequence. Keyfitz (1977: 168), in discussing shortcomings in the fit of the Lotka, Wicksell, and Hadwiger curves, wrote that "tests of the three graduations offer little room for satisfaction," but he added, "nor are the consequences of poor fit always serious."

Figure 3: Plots of observed and cubic five-year birth rates [φ (x,5)] for the United States, 2020



4. Analytical applications of cubic fertility

Here we consider three areas of application: (1) the implications for cohort fertility of a constant period fertility level (quantum) accompanied by an upward shift in the mean age of fertility (tempo); (2) examining how population momentum is affected by the stationary population mean age of fertility; and (3) outlining ways the cubic relationship can be extended to birth trajectories over time.

4.1 Cohort fertility under constant quantum and changing tempo

Ryder (1964) introduced the concept of period/cohort translation – that is, the relationship between period and cohort fertility when age-specific birth rates change over time. The topic was further explored by Bongaarts and Feeney (1998), Zeng and Land (2002), and Schoen (2022). While the quadratic ASBR curve used in Schoen (2022) led to useful new translation expressions, that curve has a fixed mean age of fertility and hence cannot examine the implications of changing tempo. As those studies demonstrate, it is not a simple matter to change period tempo while holding quantum constant and to do so in a demographically reasonable, analytically tractable, and substantively realistic manner.

Here we do so by introducing a new approach that makes use of the ability of cubic ASBRs to allow changes in period tempo while holding period quantum constant. We begin at Time 1 with cubic ASBRs at replacement level (R = 1) and a mean age of fertility of $\mu = 27$ years. Keeping R constant, we let μ increase by 0.4 years every five-year time interval, until $\mu = 31$ at Time 11. Before Time 1, μ is always 27 and after Time 11, μ is always 31. Using the cubic expressions for ASBRs in Equations (5) and (15), we can proceed to examine cohort net reproduction rates (CRRs) and cohort mean ages of fertility (μ_c). Table A-1 shows the array of period ASBRs for Times 1 through 11. Those period rates are turned into cohort rates by simple averaging – that is, the cohort fertility rate at ages x to x + 5 for the cohort born in period t is given by

$$\varphi_c(x,5,t) = [f(x, t+x/5) + f(x, t+(x+5)/5] / 2, \tag{16}$$

where time moves in units of five years. Summing those cohort rates and calculating their average ages yields the CRR and μ_c values shown in Table 4.

Table 4: Cohort net reproduction rates and mean ages of fertility with period net reproduction fixed at 1 and period μ increasing by 0.4 years at every five-year interval, from 27 to 31 years

Cohort year of birth	CRR	Cohort mean age μ_c	$\Delta\mu_c$ $[\mu_c(t+1)-\mu_c(t)]$	Ryder adjusted NRR $[NRR/(1-\Delta\mu_c/5)]$
–8	1	27	–	–
–7	1.004	27.062	.060	1.012
–6	1.018	27.241	.179	1.037
–5	1.040	27.486	.245	1.052
–4	1.062	27.733	.247	1.052
–3	1.076	28.002	.269	1.057
–2	1.080	28.335	.333	1.071
–1	1.080	28.705	.370	1.080
0	1.080	29.075	.370	1.080
1	1.080	29.445	.370	1.080
2	1.080	29.815	.370	1.080
3	1.076	30.141	.326	1.070
4	1.062	30.386	.245	1.051
5	1.040	30.592	.207	1.043
6	1.018	30.795	.203	1.042
7	1.004	30.947	.152	1.031
8	1	31	.052	1.010
Bongaarts–Feeney adjusted NRR = NRR/(1−Δμp) = 1/(1−.4/5) = 1.087				

Note: Period mean age of fertility is 27 at Year 1 and rises to 31 in Year 11. The symbol μ_p denotes the period μ. Cubic ASBRs for each year are shown in Table A-1. For calculation procedure, see text.

The CRR goes from 1 for those born at Time −8 to a maximum of 1.08 for cohorts born at Times −2 through +2, before falling back to 1 for cohorts born at Time 8 and after. The cohort mean age of fertility rises monotonically but unevenly from 27 to 31, with the increase in μ_c greatest for the cohorts born at Times −1 through +2. A rising and then stabilizing period of mean age of fertility thus produces a temporary increase in cohort fertility, even though there was never an actual rise in the level of period fertility. While Ryder (1964) emphasized that cohort tempo can affect period quantum, here we see that period tempo can impact cohort quantum. Demographers have debated the primacy of cohort vs. period in fertility analysis, with Ní Bhrolchaín (1992) vigorously disputing Ryder's cohort-centric view. Here we present an analytical, not a theoretical, exercise. By taking period tempo change as the independent variable, we demonstrate the consequences of that change for the completed fertility of the cohorts affected. Ryder (1964) further argued that R values for time period t can be translated to CRR values for the cohort born at time $t-\mu_c$ by the relationship

$$CRR(t-\mu_c) = R(t) / (1 - \Delta\mu_c/5), \tag{17}$$

where $\Delta\mu_c$ is $(\mu_c(t) - \mu_c(t-1))$. The last column of Table 4 shows Ryder's estimated CRRs, which are accurate for the mid-era birth cohorts but off for earlier and later cohorts.

The calculated cohort values in Column 2 of Table 4 can also be compared to the estimates offered by Bongaarts and Feeney (1998), who proposed the translation relationship

$$CRR(t-\mu_p) = R(t) / (1 - \Delta\mu_p/5), \tag{18}$$

where μ_p (aka μ) is the *period* mean age of fertility. Here $\Delta\mu_p/5$ is constant at $0.4/5 = 0.08$, and R is constant at 1. The Bongaarts–Feeney estimated CRR is thus always $1/.92 = 1.087$. That is a rather poor estimate, especially for the first and last five cohorts.

In short, the ability of the cubic parameterization to separate fertility quantum and tempo allows a straightforward calculation of the effects of changing period tempo in the presence of constant period quantum. The results show that previous translation estimates of the CRR in such a scenario are subject to considerable error. Table 4 not only provides new results regarding period/cohort translation; it also presents a significant new approach. The cubic parameterization, by allowing an analysis to separately and smoothly manipulate both period and cohort values, gives analysts an important new tool for examining population dynamics.

4.2 Assessing the influence of fertility timing on the level of population momentum

Population momentum reflects the extent of population growth that occurs after a growing population attains replacement level net maternity (R = 1). Keyfitz (1971) introduced the concept and derived an expression for momentum: the ratio of the size of the ultimate stationary population to that of the initial population, assuming an initial stable population and an immediate proportional drop in fertility to replacement level.

Here we want to examine a somewhat more general case, where fertility can fall in any manner but where the zero growth ASBRs follow a cubic curve. In zero mortality models extending only up to the highest age of reproduction, that generalization is not difficult. Scaling the initial number of births to one, the number of persons in the j^{th} five-year age group, j = 1…9, is exp[−r(5j−2.5)]. Thus the population in the first age group is exp[−2.5r]. Under the cubic assumption, the replacement level population projection matrix (PPM) for any given μ can be found from Equations (5) and (15). That zero-growth PPM, raised to the 100^{th} power and multiplied by the initial population vector, yields the ultimate stationary population. The 100^{th} power was chosen to insure that the PPM was in rank 1 form; that, is the product of a column vector of 1s and a row vector of reproductive values (see Schoen 2006: 28, 157 (2b)). Any sufficiently large exponent would yield the same stationary population.

Table 5 shows the momentum values that arise from μ values of 27, 30, and 33, combined with initial stable population growth rates of −0.01, 0.01, 0.02, and 0.03. The level of the initial r is clearly the stronger influence, but different levels of fertility tempo produce differences in momentum of some 3% to 7%. Table 5 quantifies those tempo effects for the first time. Since differences of that size may well be noteworthy, the ultimate mean age of fertility does merit attention in momentum calculations.

Table 5: Population momentum as a function of stable growth rate r and ultimate cubic mean age of fertility μ

Ultimate mean	Momentum for r =			
Age μ	−.01	.01	.02	.03
27	.918	1.081	1.157	1.229
30	.933	1.065	1.126	1.182
33	.945	1.052	1.100	1.143

4.3 Extending the cubic parameterization to time-varying birth models

The birth model described here has its tempo from cubic fertility density f(x), set forth in Equation (5), and its quantum from fertility level R, which adjusts fertility rates proportionately at all ages. The dynamic extension of the model has level R(t) varying over time and allows changes in mean age μ. Two different analytical contexts should be considered.

In the first context, the R(t) trajectory and the value(s) of the mean age of fertility [μ(t)] are known. The birth trajectory and all the time-specific ASBRs can then be found from the specifiable population projection matrices. In the second context, the birth trajectory and the μ(t) trajectory are known. The R(t) values, and hence the ASBRs, can then be determined from the projection equation for each time interval.

Using such dynamic models, relationships between fertility levels and birth sequences can be explored for a broad range of trajectories. The cubic parameterization enables such analyses to examine in depth the distinct roles of fertility quantum and fertility tempo, which can be of value as work continues to explore the dramatic changes in fertility that have characterized recent decades.

5. Summary and conclusions

The cubic parameterization provides a useful summary of fertility curves. For analytical purposes, it provides an adequate representation, and the model provides new methodological flexibility by separating fertility tempo and quantum. In particular, the ability to vary fertility timing while holding fertility level constant is a powerful new tool for analyzing pure timing effects and provides new period/cohort relationships. The cubic also affords other analytical advantages, including (1) a new method for approximating Lotka's r and determining its sensitivity to the level of fertility, (2) a method to gauge timing influences on population momentum, and (3) a structure for relating fertility and birth sequences in dynamic models.

The cubic parameterization does not always provide a good fit to observed ASBRs, as it requires a cubic fertility density and cannot reflect the observed variance of the ASBRs. While cubic expressions such as Equation (5) are a bit lengthy, the rationale behind them is simple and the potential applications are quite extensive. The cubic parameterization deserves serious consideration from fertility analysts.

6. Acknowledgments

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Appendix A. Deriving the cubic fertility density function

Let the base age curve of fertility be given by the cubic equation

$$f(x) = a + bx + cx^2 + dx^3 \quad \text{(A.1)}$$

To determine the four cubic coefficients (or parameters), one must solve a set of four equations. Three of them are structurally specified, as the curve must equal 0 at age 15 and age 45, and the area under the curve between those ages must equal 1. Hence there is only *one* choice parameter, which can be taken to be mean age μ. The specifying equations are then

$$\begin{aligned} f(15) &= 0 = a + 15b + 225c + 3375d \\ f(45) &= 0 = a + 45b + 2025c + 91{,}125d \\ 1 &= \int [a + bx + cx^2 + dx^3]\, dx \\ \mu &= \int x\, [a + bx + cx^2 + dx^3]\, dx \end{aligned} \quad \text{(A.2)}$$

Integrating in the third equality of Equation (A.2) yields

$$1 = 30a + 900b + 29{,}250c + 1{,}012{,}500d \quad \text{(A.3)}$$

Carrying out the integration in the fourth equality of Equation (A.2), we find

$$\mu = 900a + 29{,}250b + 1{,}012{,}500c + 36{,}753{,}750d \quad \text{(A.4)}$$

Assuming that mean age of fertility μ is known, we can use the four equalities from Equation (A.2) to solve for the cubic parameters in terms of μ and find that

$$\begin{aligned} a &= (-63/20 + \mu/10) \\ b &= (19/50 - 11\mu/900) \\ c &= (-61/4500 + \mu/2250) \\ d &= (1/6750 - \mu/202{,}500) \end{aligned} \quad \text{(A.5)}$$

Equation (A.5) allows the cubic f(x) to be expressed as a linear function of μ:

$$\begin{aligned} f(x) = &(-63/20+\mu/10) + x(19/50 - 11\mu/900) + x^2(-61/4500 + \mu/2250) \\ &+ x^3(1/6750 - \mu/202500) \end{aligned} \quad \text{(A.6)}$$

Equation (A.6) provides the desired cubic density function.

Appendix B. Finding the peak age of the cubic f(x) curve

Equation (5) expresses the cubic f(x) as a function of mean age of fertility μ:

$$f(x) = (-63/20+\mu/10) + x(19/50 - 11\mu/900) + x^2(-61/4500 + \mu/2250) + x^3(1/6750 - \mu/202500) \quad \text{(B.1)}$$

To find the peak age of the f(x) curve, we differentiate with respect to age and set the derivative equal to 0. Hence

$$df(x)/dx = (19/50 - 11\mu/900) + 2x(-61/4500 + \mu/2250) + 3x^2(1/6750 - \mu/202500) = 0 \quad \text{(B.2)}$$

The peak varies with μ but can be found from quadratic Equation (B.2) for any value of μ. Table 1 shows peak ages for mean ages of fertility 27, 29, 30, 31, and 33.

Appendix C. Deriving the skewness of the cubic f(x) curve

The cubic age curve of fertility, at replacement level, is given by Equation (5). By definition, the skew of the f(x) distribution, the third moment about the mean, can be written

$$\text{Skew} = \int (x-\mu)^3 f(x)\,dx, \quad \text{(C.1)}$$

where the integral ranges from 15 to 45. Hence the skew is the integral of the product of two cubic functions and can be found by basic calculus and straightforward algebra.

To begin, expand the cubic expression and write

$$\begin{aligned}\text{Skew} &= \int [x^3 - 3x^2\mu + 3x\mu^2 - \mu^3]\, f(x)\,dx \\ &= \int x^3 f(x)\,dx - 3\mu \int x^2 f(x)\,dx + 3\mu^2 \int x\, f(x)\,dx - \mu^3 \int f(x)\,dx \quad \text{(C.2)}\end{aligned}$$

From Equation (7), the second integral in the last equality of Equation (C.2) is $(\text{Var}+\mu^2)$, where Var in terms of μ is given in Equation (8). The third integral in the last equality is μ, and the fourth integral is simply 1. Finding the skew thus requires finding the first integral, which we can denote as Z_3, as it is the third moment of f(x) about 0.

Proceeding by multiplying x^3 by the cubic expression in Equation (5) and integrating from age 15 to age 45 using the "int" function in Maple yields

$$Z_3 = -52{,}842.865 + 2796.429\mu \quad \text{(C.3)}$$

Combining terms in the context of Equation (C.2) gives the result

$$\text{Skew} = 2\mu^3 - 180\mu^2 + 5361.429\mu - 52{,}842.865 \qquad \text{(C.4)}$$

Table 1 shows values of the skew of f(x) for mean ages of fertility 27, 29, 30, 31, and 33.

Appendix D. Discrete age-specific fertility values [φ(x,5)] and Leslie (projection) matrix first-row values [F(x,5)] associated with the cubic f(x) density function, expressed in terms of mean age of fertility μ

The continuous form of the f(x) density function in Equation (5) can readily be transformed into discrete values for applied work. Specifically, for the five-year interval from age x to age x + 5, the discrete fertility rate φ(x,5) can be found from

$$\varphi(x,5) = \int f(x)\, dx, \qquad \text{(D.1)}$$

where the integral ranges from x to x + 5.

Performing that integration, we find

$$\varphi(x,5) = (-11.5417+.3650\mu) + x(1.5796-.0506\mu) + x^2(-.0622+.0020\mu) + x^3(.00074-.0000247) \qquad \text{(D.2)}$$

Values of φ(x,5) for ages 15–19 through 40–44, in terms of mean age μ, are shown in Table 3. Values of the first-row elements of a Leslie matrix can then be found from Equation (15), as described in the text, and are also shown in Table 3.

When fertility level R is fixed at 1 but μ increases by 0.4 years every five-year time interval for ten five-year intervals, the *cohort* values for φ(x,5) can be found from Equation (16). The array of age-time-specific fertility rates, as discussed in Section 4.1 and as rearranged and summarized in Table 4, is presented in Table A-1.

Table A-1: Discrete cubic ASBRs with R = 1 and mean age μ going from 27 to 31 years by 0.4 years every five-year interval for ages 15–19 through 40–44

	Year											
	1	2	3	4	5	6	7	8	*8.5*	9	10	11
Age \ (μ)	(27)	(27.4)	(27.8)	(28.2)	(28.6)	(29)	(29.4)	(29.8)	*(30)*	(30.2)	(30.6)	(31)
15–19	1319	.1242	.1165	.1088	.1011	.0934	.0856	.0779	*.074074*	.0702	.0625	.0548
20–24	.2755	.2634	.2514	.2393	.2273	.2153	.2033	.1912	*.185185*	.1792	.1671	.1551
25–29	2801	.2749	.2696	.2643	.2591	.2538	.2486	.2434	*.240741*	.2381	.2329	.2276
30–34	.2014	.2066	.2119	.2172	.2223	.2276	.2329	.2381	*.240741*	.2434	.2486	.2538
35–39	.0949	.1070	.1190	.1310	.1431	.1551	.1671	.1792	*.185185*	.1912	.2033	.2153
40–44	.0162	.0239	.0316	.0394	.0471	.0548	.0625	.0702	*.074074*	.0779	.0856	.0934
Total	1	1	1	1	1	1	1	1	*1*	1	1	1

Note: Reproductive ages range from 15 to 45.